namely, that one quantity may proceed in arithmetic progression as another related quantity proceeds in geometric progression, is ordinarily not brought out, nor is the fact that this same situation leads to a logarithmic law.

The omission of these and similar vital connections, both of mathematics to the exterior world and of one topic in mathematics to another, is directly responsible for the failure of algebra to reach the hearts of our students, and for the failure of the students to gain real insight into the significance of the subjects they so dully learn.

I shall not dwell long on any one topic, for I desire to emphasize the existence of significance for life and society in the entire range of mathematical courses, and I desire to call your attention to the failure—shall I not say *our* failure?—to bring to light that significance.

Let me turn to analytic geometry for another instance of our traditional blindness, if it be that—our sin, if it is not blindness. Here, as before, applications abound. Most of the results of scientific experiment today are known and are recorded not by algebraic formulas of traditional form, but solely by curves traced in our traditional style, showing graphically the functional relations between two or more interdependent variables. Laws of physics, of chemistry, of every quantitative science, expressed by such means abound. The effort of science may well be said to be to deduce from such graphical functions the corresponding laws in algebraic or formalistic form.

Yet to most students of analytic geometry, precisely the reverse view seems to be our aim. The significance of analytic geometry as a piece of scientific machinery is totally lost, and the subject sinks to the level of dubious value in the minds of our students and of half-informed educators. In the present emergency, popular conviction of the real significance of analytic geometry for society is being attained, and may be fostered, through the occurrence of just such graphical laws in the dynamics of airplanes, in artillery performance (ballistics) and in wireless telegraphy. Here, as in general in science, most of our information on functions is now in graphical form, and the desire to express the function in equation form illustrates the fundamental demand of science, and the fundamental significance of analytic geometry.

That the calculus is regarded as dry and uninteresting by many students, and that its value is occasionally doubted, is the strongest proof possible that its significance is not grasped. Here the connection with realities is so easy and so abundant that it is actually a skillful feat to conceal the fact. Yet it is done. I know personally of courses in the calculus (and so may you) in which the pressure to obtain and to enforce memory of formal algebraic rules has resulted in absolute neglect of the idea that a derivative represents a rate of change! I know students whose whole conception of integration is the formalistic solution of integrals of set expressions by devices whose complexity you well know. That an integral is indeed the limit of a summation, and that results of science may be reached through such summation is often nearly ignored and not at all appreciated. That the ideas of the Calculus should fall so low as to consist mainly in

formal differentiation and integration of set expressions must indeed astound anyone to whom the wonderful significance of the subject is at all known. Moreover, it must convince any liberally minded educator who takes our own courses as a true representation of mathematical values that even the calculus is of no importance for real life or for society.

I might proceed to other courses—differential equations as given by Forsyth, the theory of equations as by Burnside and Panton or as by even the most recent writers, the theory of functions (without any hint of its manifold connections with physics), the calculus of variations (denatured, without a hint of its vast importance in mechanics and elsewhere), projective geometry (with no mention of descriptive geometry nor of the representation of space forms).

In all these, tradition has been leading us as far astray as it has in those more elementary courses of the secondary school, which we are wont to criticize. Shall we not search our own house? Shall we not ask if our own collegiate and graduate courses in mathematics demonstrate to students the real significance of the theory they cover? Have we denatured each subject until insight is eliminated and only formalism and logical tricks remain? So long as this blight remains, we must expect and we shall deserve public disdain and sincere doubt of our value to humanity.

It should be unnecessary for me to explain my own deep interest in the logical and cultural side of mathematics. Certainly I would be the last to belittle its great spiritual values. But this is for the specialist rather than for the usual student. Values to the world at large must be stated in terms of more concrete realities. Shall we hide the fact of the immense service of mathematics to society? To emphasize beauty and pleasure to the entire exclusion of the more convincing argument of benefit to mankind is as quixotic and short-sighted as is the corresponding formalization of our courses of instruction. To ignore the significance of our great subject is to spurn our birthright.

Let me then, in retiring from office in the Association, leave with you the sincere hope that a part of the work of this Association may be to impress upon the public the great value of mathematics in its direct effects upon life and upon human society. To accomplish this end, a most effective means, and one ready to hand, is to bring out to our own students, not halfheartedly, but with vigor, not a few but all available facts that shed light on the real meaning of what we teach. Let this Association be a focus from which such doctrine may emanate, a forum in which such views may be emphasized and detailed. Thus I today have mentioned to you a few samples of our neglect, in haste and by name only. Shall we not discuss among ourselves these and other means toward the end, other topics whose significance is commonly lost or neglected, other points of view that will increase insight, even if it be at the expense of a few formulas or theorems that we traditionally treasure.

To the same end, may I now emphasize what seems to me a great, if not the greatest, function of this Association. In America, up to recent years, the beauty and interest centering in pure mathematics has so absorbed all mathematical

talent that we have almost if not quite neglected that other phase of mathematics in which the significance of all we do is so self-evident: applied mathematics. This Association has, through its journal and through its meetings, already demonstrated its willingness and its ability to foster mathematics of this type. On this side of mathematics, not only discussion of the mathematics taught or to be taught, but even research papers of high grade have had in the past no adequate means of exposition. The wonderful work of Gibbs was for this reason long buried in the obscure Connecticut Academy, and mathematical advancement along the important lines that he laid down was delayed or wholly prevented. The great work of G. W. Hill, which included profound work on infinite determinants, was for the same reason unknown and unappreciated by many mathematicians in this country until near his death, and work by others along the lines he mapped out was discouraged and delayed. Thus American mathematics has suffered not only in reputation, through the suppression of what are perhaps the greatest American achievements in mathematics, but also in that encouragement necessary to the establishment of a strong school. The same may be said of the essentially mathematical researches of other men still living, whom I hesitate to name,[1] whose work is scattered through journals on general science, journals on astronomy, journals on life insurance, journals on engineering, and so forth.

Already there have been published by the MONTHLY articles of research on topics in insurance, on mathematical history, on mechanics, and on other applied branches of mathematics. In the first annual meeting, Professors Wilson and Webster presented their own studies on the mathematical theory of the dynamics of the air. At the last summer meeting Professors Huntington and Hoskins presented studies on the foundations of mechanics. Such work, though deserving of high praise, has long had no suitable center for exposition and for encouragement. This Association has afforded a means for exposition, as within the field of mathematics in its broader meaning, of papers in applied branches of mathematics. I trust that we shall continue this policy, and that we shall no longer rule out of our circle in mathematics those who find the problems of applied mathematics peculiarly attractive. It should be our aim to encourage them and their students; to hear their work and to print it; to listen to their counsel on the needs of our traditional mathematical courses; to learn from them ourselves to appreciate more keenly the significance of mathematics as a whole.

In both these ways,—by reorganizing our own instruction under the auspices of this Association, and by the recognition and encouragement of workers in the various fields of applied mathematics, we may, and I think we should, increase the appreciation of the significance of mathematics among our students, among the public, and even among ourselves. Incidentally we shall have done a service, not only to the public, in the increased emphasis upon phases of mathematics of real public service, but also to the advancement of mathematics itself, in

[1] One of the men I have in mind is in attendance at these meetings.

that a better insight into the significance of mathematics will prevent or nullify mistaken attacks on the subject as one of little public worth.

Such to my mind should be one function, if not the chief function, of this Association: the regeneration of a significant mathematics, the encouragement of workers in applied mathematics, and the effort to obtain recognition of the true public worth of mathematics in every phase.

THE LINEAR FUNCTION AND THE LINE.

By JOSEF NYBERG, Chicago, Ill.

In a previous paper[1] I showed how the work in college algebra, trigonometry, and analytics may be unified by treating the contents of these subjects under the general theory of functions. In a second paper[2] I then showed how the notion of function should be introduced so that the student may see that coordinates constitute only the language of mathematics, the subject itself dealing with the properties of functions and relations between variables. The present paper begins the study of the line, and closes with some illustrations of how the new point of view eliminates certain difficulties.

I begin the work by stating that the rate of increase of a function is one of its most important properties, and that the simplest function is that which has a known constant rate of increase. For such a function we may define the rate as the ratio of the change in the value of the function to an increase in the value of the variable. We can measure this rate by the change in the function per unit increase in the variable. This idea is impressed by writing some data on the board, as

$$\begin{cases} x & -1 & 0 & 1 & 2 & 3 \\ y & 2 & 5 & 8 & 11 & 14, \end{cases} \qquad \begin{cases} x & -1 & 1 & 3 & 5 \\ y & 4 & 9 & 14 & 19, \end{cases}$$

and asking such progressive questions as: What is the change in x? in y? What is the rate of increase? What is the value of y for $x = 0$? How can the value of y be computed for a value of x not given in the data? When the student has seen that the general relation is $y = y_0 + mx$ the more formal work may be presented in the following order.

I. Assuming that the function has a constant rate of increase m, and that several corresponding values are known, we prove that the graph of the function is a line; and conversely, a line is the graph of a function with a constant rate of increase. The proof naturally hinges on the similarity of certain triangles; and as these triangles will be similar only when m is constant, the student will associate a line with a uniform rate. I prefer the form $y = y_0 + mx$ to the form $y = mx + b$, because the physicists and engineers are accustomed to such relations as $v = v_0 + at$, $s = s_0 + vt$, $p = p_0 + (\rho g)h$ and because $y = y_0 + mx$

[1] "The Unification of Freshman Mathematics," in this MONTHLY, April, 1916, page 101.

[2] "The Presentation of the Notion of Function," in this MONTHLY, September, 1917, p. 309.

is the only statement of how the student actually does compute any value of y in a numerical case: he adds to y_0, a reference value, the amount mx, which is proportional to the increase in x from 0, another reference value. Even if we are dealing with an abstract relation, b would not be as good a symbol as y_0 because it is entirely lacking in connotation.

II. As a second step we show formally that every linear relation $Ax + By + C = 0$ $(B \neq 0)$ can be written as $y = y_0 + mx$ and that its graph is a line.

III. If the value of y_0 is not known, but the rate of increase and a pair of corresponding values are known, we do not need to compute y_0 explicitly, as the student did in the preliminary exercises, but can show that the relation is $y = y_1 + m(x - x_1)$. Further, if the rate of increase is unknown the student will see that the function is undetermined unless we know several corresponding values from which the rate can be found as

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The work thus emphasizes that a pair of values and a rate determine a function, and does not particularly emphasize such an equation as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Even the discussion of an abstract line can be based on the form

$$y = y_1 + m(x - x_1)$$

for it can be used to derive the "two intercept" form and the "perpendicular" form, wherein a point $(p \cos \alpha,\ p \sin \alpha)$ and a slope $(-\cot \alpha)$ are known. I do not consider here the other problems dealing with families of lines or locus problems, as I am concerned only with the introduction to the subject.

There are two points in this method to which I call attention. The first is that all the beginning work is grouped around a central idea—the study of a function determined by a value and a rate of increase. This idea unifies the separate problems. Examining the corresponding treatment of lines in any textbook, we note the great diversity even in a single book in the modes of attack on these problems. Such problems as the line through two points, the point-slope form, the perpendicular form, the determination of the slope from the equation, the condition that lines be parallel or perpendicular, all these appear as *distinct* problems, completed only when a formula is found for writing the answers to analogous problems. Thus, the practice of using an equation like

$$\frac{y - y'}{y' - y''} = \frac{x - x'}{x' - x''},$$

found in several books, is especially pernicious in that it has absolutely nothing in common with other forms, and because the simpler concepts on which the

concept of a line is built are wholly missing. Again, the use of projections in finding the perpendicular form of a line is undesirable because it seems to the student to be only another trick of the omniscient teacher. By the present method, however, finding the perpendicular form of a line involves no trick, for the various forms are based on one as fundamental. Similarly the equation of a line through two points is easily derived by the student when he sees the data to be only a bit of information for determining the slope. Although we do not ordinarily ask the student to discover such relations, the fact that he can is a test of his understanding and a proof that he is thinking, not memorizing. When the problems do not center around an idea the student is bound to resort to memorizing.

Secondly, I wish to remark on the pedagogical correctness of the procedure. We know that the permanence of an idea depends on the interest with which it was first aroused and the attention which that interest stimulated. Psychology thus requires that we begin with as concrete a problem as possible, and that we base it on the student's previous knowledge. The student is familiar with the notion of rate. Every day he has determined the position of some object, a train, a car, or a person he wishes to meet, by knowing the rate of motion and the position at some moment. Further, concreteness in its broad sense means not only that we begin with a study of $3x + 4y + 5 = 0$ instead of $ax + by + c = 0$, but that we begin with some very definite question about the line $3x + 4y + 5 = 0$, and that this question be fundamental, not accessory. Certainly the slope of a line is more significant than its intercepts, or the points that it may pass through, or the appearance of its graph.

As an illustration of the usefulness of the phrase "relation between variables," consider the problem of finding the coordinates of a point P_0 dividing a segment $P'P''$ in a given ratio and leading to the equation

$$x_0 = \frac{k_1x'' + k_2x'}{k_1 + k_2}.$$

To the student this is a formula for computing x_0 when P' and P'' are given, and he experiences no difficulty with it except in two cases: (1) when $P'P''$ is divided externally, and (2) when P_0, P', and P'' are given and the ratio is to be found. On the other hand, given two points (2, 3) and (12, 8), and asked to find a point three fifths of the way between them, the student will compute directly that the desired point is $2 + \frac{3}{5}(12 - 2), 3 + \frac{3}{5}(8 - 3)$. Hence, calling the points P_1, P_2, P_3 we have $x_2 = x_1 + \lambda(x_3 - x_1)$ where λ is a certain ratio. The student should never hear that this is a formula for computing x_2 but should be taught that it is a statement of a relation between four quantities x_1, x_2, x_3, λ. *Any three* of the quantities being known, this relation enables us to determine the *fourth*. The notation $P_0P'P''$ attracts undue attention to P_0, while the notation $P_1P_2P_3$ puts all the points on an equal footing giving merely the relation between them. This point of view will eliminate the second difficulty mentioned above. The first difficulty is treated by always lettering the points from left to right

$P_1P_2P_3$ (irrespective of which is the unknown), determining λ from the data (if it is not given explicitly), and then applying the relation. I have found that the complete elimination of the word "formula," in terms of which the student likes to think, and the substitution of the idea that the equation is a statement of a relation between various quantities is very helpful in all such problems.

A similar situation is met when dealing with the angle between two lines. The student can use the formula

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

to find $\tan\theta$, but the problem of finding a line making a given angle with a given line is more difficult. This trouble, like the previous one, is avoided by impressing the fact that $(1 + m_1m_2)\tan\theta = m_1 - m_2$ is a relation between three quantities, m_1, m_2, and $\tan\theta$, any two of which being known the relation enables us to determine the third. It is not a mere change of terminology which is used for removing the difficulty; we are giving the student a better conception of the word *equation*.

THE GRAPH OF $F(X)$ FOR COMPLEX NUMBERS.

By A. F. FRUMVELLER, Marquette University.

Teachers of analytic geometry are often asked why the complex roots of $f(x)$ cannot be diagrammed. The answer usually given, that real points of the graph lie in the plane of the blackboard, while complex points are in front of this plane or behind it, is true enough; but it leaves the student unsatisfied. He is not able to carry out the idea for himself; and the teacher may not have thought of working it out, nor can he lay his hand on any systematic treatment of the matter. Yet the topic is important, not to say interesting, since several lines of thought here converge: a rather full development with ample illustrations may consequently be welcomed. The purposes of a mathematical club, or undergraduate seminar, would be beautifully served by a handling of graphs along the following lines. Care has been taken with the notation: x invariably stands for $u + iv$; y stands for $U + iV$; complex constants are designated by Greek letters; thus, $\alpha = a + iA$, $\rho = r + iR$, etc. We restrict $f(x)$ to algebraic types merely for the sake of uniformity in the presentation.

The literature on this subject is so slight that the writer is able to mention only three sources to which the student may go for further information; these are:

1. Schultze's "Graphic Algebra" (The MacMillan Company, 80 pp.): methods are given for the construction of the complex roots of quadratics, cubics, and quartics, the purpose being merely to find these roots, and not to exhibit the complete graph.

2. Hamilton and Kettle's "Graphs and Imaginaries" (Longmans, Green and Company, 40 pp.): the quadratic alone is treated. Complex roots, imaginary

intersections of two circles, tangent-points of a tangent to a circle from a point within, are constructed most elegantly by means of the polar and an auxiliary conic called the "shadow-curve." The method is not available for general graphing.

3. Phillips and Beebe's "Graphic Algebra" (Henry Holt and Company, 157 pp.): the last 60 pages deal with the complete complex graph of the reduced quadratic, cubic, and quartic, but only when their coefficients are real. Triangular or 60°-paper is used, and the complex branches are plotted in the usual way, point by point, but with many skilful simplifications. The methods of the present paper are not employed, so that the student should by all means consult this book in connection with the following section. See also the article on "Imaginaries in Geometry" by E. W. Davis, in *Nebraska University Studies*, volume 10.

§ 1. *X* Complex, *Y* Real.

When x is real, its domain or field is the line OX, and real values y_k of the function $y = f(x)$ are represented by perpendicular ordinates erected at x_k on OX. If now x belongs to the number-system $u + iv$, its field is the plane uov; and just as before, we associate with every point (u, v) of the plane a real perpendicular ordinate y. As x ranges over its plane, the values of $y = f(u + iv)$ will sometimes be real, sometimes complex; the real ordinates will be found standing on certain curves $u = \varphi(v)$, whose equations can at once be found. For example:

(*a*) $y = mx + n$. Since $y = (mu + n) + imv$, real ordinates occur only when $v = 0$, *i. e.*, they stand on the axis of reals, and there are none anywhere else in the plane. The plane uoy is of course identical with the plane YOX of ordinary analytic geometry—a fact to be remembered throughout.

(*b*) $y = kx^2 + mx + n$. When $x = u + iv$, we find

$$y = (ku^2 - kv^2 + mu + n) + iv(2ku + m).$$

If y is to be real, either $u + (m/2k) = 0$, or $v = 0$, so that all real ordinates stand on two straight lines in the x-plane; namely the axis of reals, and a perpendicular to it at $[-(m/2k), 0]$. When $v = 0$, we have $y_1 = ku^2 + mu + n$ as the equation of that branch of the graph lying in the plane uoy; when $u = -(m/2k)$, $y_2 = [n - (m^2/4k)] - kv^2$ is the other branch, lying in a plane parallel to voy. The complete locus consists, therefore, of two parabolic branches of opposite curvature in perpendicular planes with a common vertex-ordinate at $[-(m/2k), 0]$, whose length is $n - (m^2/4k)$. One branch never meets the x-plane; the other pierces it in two points, whose coördinates are precisely the two complex, or the two real, root-values of $f(x)$, according as the discriminant is minus or plus.

To make this clearer, take $y = x^2 - 3x + 5$ (Fig. 1).[1] On $v = 0$, and $u = \frac{3}{2}$, stand the real y's, defining the curves $y_1 = u^2 - 3u + 5$, and $y_2 = \frac{11}{4} - v^2$. This latter pierces the x-plane at $u = \frac{3}{2}$, $v = \pm\sqrt{11}/2$,

[1] The drawings were made by Mr. Andrew Keiding, senior student at Marquette University.

i. e., at $x = \frac{3}{2} \pm (i/2)\sqrt{11}$, or at $x = OQ + iRQ$. In passing, we observe a curious relation between TQ and RQ. The minimum ordinate of $y = ku^2 + mu + n$, standing at $u = -(m/2k)$, is $TQ = (4kn - m^2)/4k = k \cdot RQ^2$; hence we have $x = OQ \pm (i/\sqrt{k})\sqrt{TQ}$.

(*c*) In the cubic $y = x^3$, the real y's stand on the 60° lines, $v = 0$, and $v = \pm u\sqrt{3}$. More generally, let $y = x^3 + mx + n$. When $x = u + iv$, we have $y = (u^3 - 3uv^2 + mu + n) + iv(3u^2 - v^2 + m)$. The real y's now stand on $v = 0$, and $v^2 - 3u^2 = m$, an hyperbola, and its axis, major or conjugate, according to the sign of m. The path-equations are equations of condition.

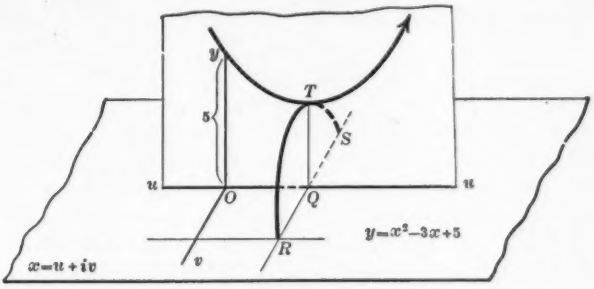


FIG. 1.

If we combine them with the original y-equation so as to eliminate v, we get as length-formulas for y, $y_1 = u^3 + mu + n$, and $y_2 = -8u^3 - 2mu + n$. These are the equations of the graphs that would be obtained by projecting the real ordinates from infinity parallel to ov onto the plane background uoy. (Similarly for the plane voy, by eliminating u.) The points on the hyperbola at which $y = 0$ give the complex roots of $x^3 + mx + n = 0$; they may be found algebraically by putting $y = f_3(u, v) + iF_3(u, v) = 0$, and solving $f_3 = 0$ and $F_3 = 0$ as simultaneous. Note that the plane uov contains only three isolated points; if $y = \sum_0^n a_k x^k = f_n(u, v) + iF_n(u, v)$, there would be n isolated points, since there exist, for every y, exactly n values $x_k = u_k + iv_k$ of x.

Nothing prevents our equations from having *complex coefficients*. Let $\alpha_1 x + \alpha_0 = y$. Then $y = (a_1u - A_1v + a_0) + i(a_1v + A_1u + A_0)$, where the real y's stand on $a_1v + A_1u + A_0 = 0$, which pierces the plane of reals in one point; hence the graph in the plane uoy is merely an isolated point. The quadratic, $(a_2 + iA_2)x^2 + (a_1 + iA_1)x + (a_0 + iA_0) = y$, has real y's on the curve $A_2(u^2 - v^2) + 2a_2uv + a_1v + A_1u + A_0 = 0$; if $v = 0$, the only points on the real axis ou, at which ordinates can stand, are given by $A_2u^2 + A_1u + A_0 = 0$ (whose roots are not necessarily real) and the length of these ordinates is furnished by $a_2u^2 + a_1u + a_0 = 0$.

In general, if $y = \sum_0^n \alpha_k x^k$, and $v = 0$, we have $\sum_0^n A_k u^k = 0$, which gives, at

most, n points on ou, the real axis; combining this with the conclusion of the previous paragraph, the result may be stated as follows:

Let $y = \sum_0^n \alpha_k x^k$. When x is $\left\{\begin{matrix}\text{complex}\\ \text{real}\end{matrix}\right\}$ and the coefficients $\left\{\begin{matrix}\text{real}\\ \text{complex}\end{matrix}\right\}$, the locus in the $\left\{\begin{matrix}\text{complex plane } uov\\ \text{real plane } uoy\end{matrix}\right\}$ consists of $\left\{\begin{matrix}n \text{ isolated points.}\\ k \text{ isolated points, } (k \leqq n)\end{matrix}\right\}$.

The condition for a real root x_r in $\sum_0^n \alpha_k x^k = 0$ is found by eliminating u from $\sum_0^n A_k u^k = 0$, and $\sum_0^n a_k u^k = 0$ (cf. Hardy, "Pure Mathematics," p. 88). The actual finding of all the real and complex roots of this equation is accomplished by solving $f_n(u, v) = 0$, and $F_n(u, v) = 0$ as simultaneous, either algebraically or graphically. The reader is advised to try the graphic solution in some simple case, such as $ix^2 + 8 = 0$, for instance. (Compare this method with that of Schultze, "Graphic Algebra," pp. 36, 51, 63.)

Implicit functions are dealt with as above, but require a little more care. Take $y^2 = x^3 = (u^3 - 3uv^2) + i(3u^2v - v^3)$; y is real when $v = 0$, and when $3u^2 - v^2 = 0$. This last gives $y^2 = -8u^3$, so that u must be negative for this part of the locus. But above all, the student should carefully construct and visualize the graph of $x^2 + y^2 - r^2 = 0$. The case of $x^2 + y^2 + r^2 = 0$ gives $y^2 = v^2 - (r^2 + u^2)$, with the condition $uv = 0$: the branch $y_1^2 = -(r^2 + u^2)$ in the plane uoy is destitute of real ordinates, but in the plane voy we have the real hyperbola $y_2^2 = v^2 - r^2$. The locus $x^2 + y^2 = i$ has no real ordinates at all. The student should carefully draw the graph of $y^2 = x^2(x - 2)$, a curve with an isolated point at the origin. Other types suitable for graphic study are furnished by x, x^2, $x^3 = f(y)$; and by y, y^2, $y^3 = \prod_0^n (x - \alpha_k)$, or $1/\sum_0^n \alpha_k x^k$, for $n = 1, 2, 3,$ employing in each equation first real, then imaginary, then complex numerical coefficients (cf. Phillips and Beebe, "Graphic Algebra," Chap. V, for a very complete treatment of the quadratic, cubic, and quartic, when the coefficients are real; many graphs are given in illustration).

§ 2. X Real, Y Complex.

At first sight, this section might seem superfluous, since on merely interchanging x and y, we come back to the preceding case. For the purpose we have in view, however, such an inversion is not allowable; we are making a clearcut distinction between the dependent and the independent variable, and while the latter ranges over its domain, the function is to be represented by an ordinate at right angles to this domain. The carrying out of this idea demands that we shall assign to x and to y, throughout this paper, an invariable meaning; the method of graphing to which we are thus led is of the highest theoretical value, since without it a clear understanding of the graph of $f(x)$, when both X and Y are complex, cannot be obtained.

The field of x is therefore a line XOX lying in space; y is of the form $U + iV$, or $y_r + y_i$. On a fixed "base-plane" UOV, perpendicular to OX, let OU, OV, be two directions at right angles for reference. These preparations having been made, let x run over its domain OX. At every point x_k we shall have a perpendicular ordinate y_k: if y_k is real, it stands out from OX parallel to OU—if imaginary, parallel to OV—if complex, at an intermediate angle depending on its amplitude, as given by the equation $y_k = f(x_k)$. The result, in general, will be a spiral curve wound around OX (Fig. 2).

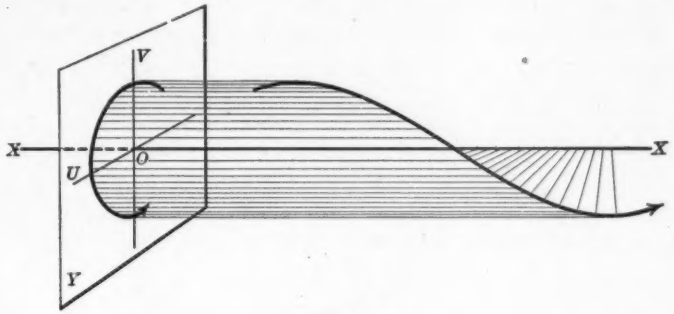


FIG. 2.

If now, we go to infinity and look back along OX, all these ordinates will be projected, unchanged in length and direction, onto the base-plane UOV, and will seem to radiate as vectors from the origin; their endpoints will map out a locus on UOV, say $U = F(V)$. This locus has its points P_k in one-to-one corre-

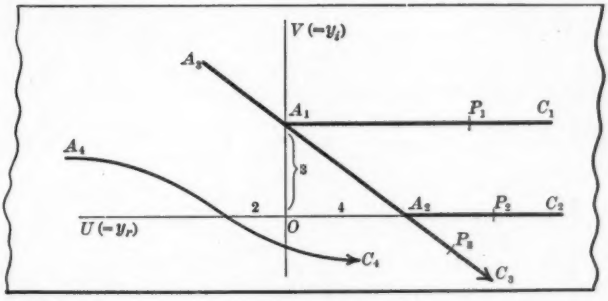


FIG. 3.

spondence with the ordinates y_k, and consequently with the values x_k of the independent variable; it is therefore in a true, though slightly unusual sense, a graph of $y = f(x)$. In reality, it is the projection of the actual locus; but the UV-graph is more compact and easier to handle; and it furnishes all the information that the actual locus could give. It is evident that the function-symbol "f" acts as an operator, transforming the segment $x_m x_n$ of the x-domain into a

segment of the curve $U = F(V)$. This manner of graphing seems to be new, although it is only an extension of the usual process. Since x is thus lost to view, its path must be depicted on a separate diagram.

To have y complex, $f(x)$ must be of the form $\psi(x) + i\phi(x)$. If $\phi(x) = 0$, the graph is altogether in the plane UOX, and its projection is simply a segment of the real axis OU, since the ordinates all pile up on top of one another; similarly for $\psi(x) = 0$. The equation of the UV-curve in parametric form will be $U = \psi(x)$, $V = \phi(x)$. The endpoint of the vector $y_k = OP_k$ is (U_k, V_k), or $(\psi(x_k), \phi(x_k))$, and its angle is $\tan^{-1}(V_k/U_k)$. Let the reader now identify the following graphs (Fig. 3), and try to visualize the space-curves of which they are the projection:

A_1C_1,	Graph of $y = 4x^2 + 3i$,	Equation;	$U = 4x^2$,	$V = 3$,
A_2C_2,	" " $y = 4x^2 + 3$,	"	$U = 4x^2 + 3$,	$V = 0$,
A_3C_3,	" " $y = (4 - 3i)x^2 + 3i$,	"	$3U + 4V - 12 = 0$.	
A_4C_4,	" " $y = x^3 - 2 - ix$,	"	$V^3 + U + 2 = 0$.	

When the equation is in the form $y = \sum_0^n \alpha_k x^k$, we obtain $U = F(V)$ very readily by Sylvester's method of elimination: thus,

$$y = \sum_0^1 \alpha_k x^k \text{ gives the straight line } \begin{vmatrix} a_1 & a_0 - U \\ A_1 & A_0 - V \end{vmatrix} = 0;$$

$$y = \sum_0^2 \alpha_k x^k \text{ gives the conic, } \begin{vmatrix} a_2 & a_1 & a_0 - U & 0 \\ 0 & a_2 & a_1 & a_0 - U \\ A_2 & A_1 & A_0 - V & 0 \\ 0 & A_2 & A_1 & A_0 - V \end{vmatrix} = 0, \text{ and so on.}$$

Among the simpler types of $f(x)$ we may note the following:

$y = a_0 + iA_0$ is the point $(U, V) = (a_0, A_0)$. The actual locus is a line parallel to OX.

$y = (b + iB)\sum_0^n a_k x^k$ is the line $\dfrac{U}{b} = \dfrac{V}{B}$.

$y = (b + iB)\sum_1^n a_k x^k + (c + iC)$ is the line $\dfrac{U - c}{b} = \dfrac{V - C}{B}$; a special type of which is

$y = \alpha + (\beta - \alpha)x$, or $\dfrac{U - a}{b - a} = \dfrac{V - A}{B - A}$, the line joining $y = \alpha$, and $y = \beta$.

$y = \dfrac{\alpha + \beta x}{\gamma + \delta x}$, a circle. (Write the complex numbers in full, clear of fractions, etc., and finally eliminate x.)

$y = \gamma + \rho\dfrac{1+ix}{1-ix}$, a circle in explicit form; on developing this expression as above, we find $(U-c)^2 + (V-C)^2 - (r^2+R^2) = 0$, so that $y = \gamma$ is the center, and $|\rho|$ the radius. (Hardy, "Pure Mathematics," p. 94.)

The reverse problem—given a graph $U = F(V)$, find $y = f(x)$—can be solved whenever U and V are expressible in parametric form. For example, $U^2 = 4V$ becomes $y = 2x + ix^2$: $U = mV + b$ gives $y = mx + b + ix$, or even $y = mx^n + b + ix^n$; $U^2 + V^2 - r^2 = 0$ becomes $y = r(\cos x + i \sin x)$, and so on. In a similar way, the value of x which makes $y = 0$, *i. e.*, which causes the graph $U = F(V)$ to pass through (0, 0), will be found by putting $U = \psi(x) = 0$, $V = \phi(x) = 0$ simultaneously, and then obtaining the H.C.F. of these expressions.

Implicit functions like $y^2 = 4px$, $y^3 = x(x-2)$, etc., are handled just as above. They have the peculiarity of furnishing both y_r and y_i when x as well as the coefficients are real. In Fig. 4 we have the complete locus of $x^2 + y^2 = 4$

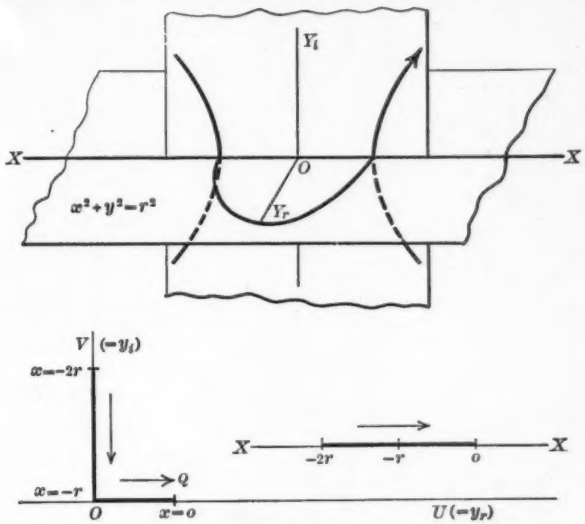


FIG. 4. Graphs of $x^2 + y^2 = r^2$.

where real ordinates lie in UOX, for $-2 < x < 2$. At $x = 2$ there is a sudden turn of 90°, but no discontinuity in the locus; the branch in VOX is $y = i\sqrt{x^2-4}$, or $y_i = V = \sqrt{x^2-4}$. The revolution of the figure about OX gives a two-sheeted hyperboloid with sphere, from which two mutually perpendicular sections through OX cut out the complete locus. The reader will note that tangents to $x^2 + y^2 = r^2$ may be drawn from points *within the circle*—a thing which at first sight would seem utterly impossible. Such tangents pass out of the two-space UOX into the next higher space. As a further exercise, one might draw the graph of $y^2 = x^2(x-2)$ for the region $0 < x < 2$.

The general case of $y^n = \psi(x) + i\phi(x)$ gives in a similar manner, at every point x_k, n different ordinates standing out from OX; a simple case for preliminary study would be $y^3 = x$, where two of the roots are complex, and one real. The arrangement of the ordinates can be followed by noting that

$$y = x^{1/3}(+1)^{1/3} = x^{1/3}\left[\cos\left(\frac{0° + n \cdot 360°}{3}\right) + i \sin\left(\frac{0° + n \cdot 360°}{3}\right)\right], \text{ etc.,}$$

by Demoivre's theorem; on UOV, the graph consists of the U-axis, and two straight lines through (0, 0) making with it an angle of 60°. Each of these lines represents a branch of this three-valued function. To indicate this clearly, we may imagine that the plane UOV consists of three transparent, infinitely thin films, not connected with one another at the origin—each film depicting the march of one branch. For the circle, Fig. 4, the situation is different. Let $y = +\sqrt{r^2 - x^2}$ be the positive branch of the function; as x travels from $-\infty$ to $-r$, the point (U, V) travels down the axis OV from $+\infty$ to 0; and thence along OU to Q, while x moves on to zero. As x moves from zero to $+\infty$, (U, V) retraces its path, thereby producing an *overlapping;* to prevent this, conceive UOV as a double film, and cut these films along OU from Q to $+\infty$; then cross-connect the films along this cut, so that the point (U, V), on reaching Q, slides down onto the second film. It must remain on this sheet till x reaches $+\infty$; and in this way, a one-to-one correspondence of the x-path, and of its transformed path in UOV, is effected. A similar cut will go from $(-r, 0)$ to $-\infty$, for the branch $y = -\sqrt{r^2 - x^2}$. (When the plane OUV is thus regarded as composed of several sheets or films cross-connected suitably, it is called a "Riemann surface.") Graphs in UOV, composed of straight-line segments, as above, parallel to the axes, can be defined only by parametric equations.

Our present method of graphing affords a clear explanation of *imaginary intersections* of curves. Take the circles $x^2 + y^2 = 25$, and $(x - 26)^2 + y^2 = 25$; they fail to meet in the plane YOX, and when solved together give $x = 13$, $y = \pm 12i$. Let us state the problem this way: "Where will these two loci meet?" Answer: "On the locus $x = 13$;" which is absolutely correct, according to Fig. 4, since $x = 13$ is a plane perpendicular to the axis OX, on which the two hyperbolic branches cross. The situation is like that in algebra, where the equation is more general than the verbal statement, and applies to conditions not explicitly foreseen. Our imaginary intersections are *actual geometric entities* in a higher space-domain,—they are answers to a wider question in a larger number-field.

In a similar way, the statement that the *polar is the chord of contact of tangents from the pole* is substantially true, even when the pole is within the locus. A circle, for example, sends out, from an inner point P on the X-axis, two actual tangents to the hyperbolic branch, and these have a chord of contact standing *perpendicular* to the axis of reals. A plane at right angles to OX through this chord leaves as its trace on the plane of the circle the line which we call the *polar* of P. The *power-axis* of two non-intersecting circles is accounted for, in the same

way, by the plane through their common chord (*cf.* Hamilton and Kettle, "Graphs and Imaginaries," ch. 5).

As a final application of the above methods, let us graph on the diagram of Fig. 4 the two curves $y^2 + x^2 = r^2$, and $Y^2 + (ix)^2 = -r^2$. On the shelf $y_r ox$, we find the loci $y_r^2 + x^2 = r^2$, and $Y_r^2 + (ix)^2 = -r^2$; on the vertical plane $y_i ox$, we find $Y_i^2 + (ix)^2 = -r^2$, and $y_i^2 + x^2 = r^2$, in the order given. Now, our $y = f(x)$ is essentially a "circular" function, and $Y = f(ix)$ is "hyperbolic." When $r = 1$, it is customary to represent the ordinates of these two classes of functions by the symbols Sin and Sinh, respectively, so that $y = f(x) = \text{Sin}\,(x)$, and $Y = f(ix) = \text{Sinh}\,(ix)$. The distances from their feet to the origin along OX are called Cos and Cosh, in like manner. The branches of our two loci are evidently congruent; hence, $\text{Cosh}\,(ix) = \text{Cos}\,(x)$, and $\text{Sinh}\,(ix) = i\,\text{Sin}\,(x)$, —a new and purely geometrical proof of the fundamental formulas of hyperbolic trigonometry.

The reader will note that if the X-axis of the graphs treated in this § 2 be identified with the real X-axis of the graphs in § 1, the imaginary Y-branches that we have discussed will lie in the fourth dimension.

§ 3. X AND Y BOTH COMPLEX.

The field of x is now the plane uov. When x was linear, its real and imaginary ordinates were at right angles, and both stood perpendicular to OX at the point x_0. The complex intermediate vectors y_c were likewise perpendicular to OX at the same point. To carry out this idea, there must now stand on every point $x_0 = u_0 + iv_0$ a real ordinate y_r and an imaginary ordinate y_i, mutually at right angles, and both perpendicular to uov. We need consequently a 4-dimensional space for our graph, y_i pointing into the fourth dimension. The plane $y_r x_0 y_i$ may be called the "absolutely" perpendicular plane at x_0; in it, radiating from x_0, lie the other complex ordinates y_c, all at right angles to the plane uov at x_0.

The easiest way to deal with this situation graphically is first to ignore one dimension, say y_i, and plot the locus in the 3-space $y_r uv$; then ignore y_r, and plot in the space $y_i uv$. A clear idea can thus be gotten of what is happening in these two directions. Real ordinates will stand on certain paths in uov, and imaginary ones likewise: thus, if $y = x^2 - 2x + 4 = (u^2 - v^2 + 2u + 4) + 2iv(u - 1) = U + iV$, real ordinates whose lengths are, respectively, $y_r = u^2 - 2u + 4$, and $y_r = 3 - v^2$, stand on the paths $v = 0$, $u = 1$. Imaginary ordinates stand on the hyperbola $u^2 - v^2 - 2u + 4 = 0$, and their lengths are given by $y_i = 2(uv - v)$. Combining the path-equation with the length-equation by eliminating u or v, we get the equation of the locus that results from projecting all the y_r's, or y_i's, onto the planes $y_r ou$, $y_r ov$, or $y_i ou$, $y_i ov$.

In this example, U and V were explicit, and the critical paths for x could be read off at first sight; but now, take the circle $x^2 + y^2 - \rho^2 = 0$. Separating the real and imaginary terms, we get (1) $u^2 + U^2 - r^2 - (v^2 + V^2 - R^2) = 0$, (2) $uv + UV - rR = 0$; no obvious x-paths present themselves. If however we assign to v the value $\psi(u)$, we can eliminate u from (1) and (2), and thus

obtain an equation between U and V. To any assigned curve $\psi(u, v) = 0$, there corresponds a definite curve $\phi(U, V) = 0$ in UOV, the "absolutely" perpendicular plane. One locus is the "transformation" of the other. The "f" in $y = f(x)$ is therefore an operator which transforms an arc of the locus $\psi = 0$ in planes 1 and 2 of our 4-space, into a corresponding arc of the locus $\phi = 0$ in planes 3 and 4. Its action is analogous to that of a mirror with a bent surface: the same function ψ, when it is transmitted through the medium of $f_1, f_2, \cdots$, comes out on the plane UOV as $\phi_1, \phi_2, \cdots$. This process is often spoken of as the *transformation* or *depiction* of the plane uov on the plane UOV.

The manner in which this depiction is brought about is as follows: Let there be given a locus $\psi(u, v) = 0$ in the x-plane, with all its ordinates, as computed from $y = f(x)$, standing upon it. Now project the plane from ∞, parallel to itself and to the line ov, onto the axis of reals ou. This leaves the *length* and the *inclination* of the ordinates y_e unchanged, because they are all perpendicular to uov, and their mutual angles are in the plane $y_r o y_i \equiv UOV$, which is perpendicular to the line ov; hence they now stand in proper length and position on the line ou. Now project this line, parallel to itself, onto the plane UOV; all the ordinates pile up against this plane and appear as vectors issuing from the origin, and their end-points map out a locus $\phi(U, V) = 0$, which is the graph of $y = f(x)$ with $x = u + iv$ as invisible parameter.

Since x has been shuffled out of sight by projecting its field uov into the point (0, 0) of the Y-plane, two separate diagrams will be needed in plotting—one, to show the path of x in its own plane—the other, to show the position and length of the vector-ordinates in the plane of Y. We may summarize the relation of these two planes as follows for explicit functions:

Let $y = f(x)$, and $x = u + iv$. To any point (u, v) corresponds a point (U, V), where $U = f_1(u, v)$, $V = f_2(u, v)$, and $y = U + iV$. To the x-path $v = \psi(u)$ corresponds the y-path $U = f_1(u, \psi(u))$, $V = f_2(u, \psi(u))$, or $F(U, V) = 0$; to the y-path $V = \phi(U)$ corresponds the x-path $f_2(u, v) = \phi[f_1(u, v)]$; it being understood that $\psi(u)$, and $\phi(U)$ may be constants. A given ordinate, $y = a + ib$, stands on the points (u_m, v_m) in the x-plane, where u_m, v_m are the common solutions of $U = f_1(u, v) = a$, $V = f_2(u, v) = b$. When $a = b = 0$, we find the complex roots of $f(x) = 0$. For instance, if $x^2 + y^2 = i$, and $U = V = 0$, we get $u = v$, and $uv = 1/2$, which meet at $(\pm (\sqrt{2}/2), \pm (\sqrt{2}/2))$ in the plane uov, where $y = 0$. We have thus found the square roots of i. From $x^3 + y = i$, we get in like manner $(0, -1)$, $(\pm (\sqrt{3}/2), 1/2)$; hence $-i$, and $(i/2) \pm (\sqrt{3}/2)$ are cube roots of i.

For implicit functions $\chi(x, y) = 0$, the separation of real and imaginary terms will yield $\chi_1(u, v, U, V) = 0$, $\chi_2(u, v, U, V) = 0$. To find the locus in UOV corresponding to $v = \psi(u)$, we must regard u as a parameter, and eliminate it, as above. If this be impossible, we can plot by points, though this is hardly ever useful in practice. Now let the reader try all this out on some simple function like $y = x^2$, $y^2 = \alpha x$, or $y = \alpha x^2 + \beta x + \gamma$—allowing x (or y) to run along lines parallel to the axes, or through the origin; or along rectangular hyperbolas, etc. The following examples afford useful practice:

(1) $y = (x - \beta)/(x - \alpha)$, when y is fixed, is the equation of a circle in uov. Write in full, separate the reals and imaginaries, solve for U and V by determinants; then

$$\frac{U}{V} = \frac{u^2 + v^2 - (a + b)u - (A + B)v + (ab + AB)}{(A - B)u + (b - a)v + (aB - Ab)}.$$

Hence, for any constant values of U and V, *i. e.*, for any fixed ordinate $y_0 = U_0 + iV_0$, we have a circle in uov as the locus.

(2) $y = \alpha x$, when α is real, effects a *magnification;* when α is imaginary, a *magnification* and a *rotation of* 90°; when α is complex, change x and α to polar form, and use DeMoivre's Theorem.

(3) $y = 1/x$, in polar form, shows us an *inversion* on the unit circle, and a *mirroring* in the real axis. What UV-curves correspond to circles in uov, center at origin? To straight lines through (o, o)? To the circle $a(u^2 + v^2) + bu + cv + d = 0$? Solve these questions for the transforming function $y = (1 + x)/(1 - x)$.

(4) $y = (\alpha x + \beta)/(\gamma x + \delta)$ transforms all circles into circles. (Compare (1) above.) It gives a straight line in UOV when $u^2 + v^2 = 1$, and $|\alpha| = |\gamma|$.

(5) $y = x + \alpha$ is merely a parallel transformation.

(6) $\alpha x + \beta y + \gamma = 0$ is an "imaginary" straight line; only one pair of real values (x, y) lies on this line. When the coefficients are real, it has an infinite number of conjugate imaginary points. When will the join of two imaginary points (lines) be real?

(7) Given two simultaneous equations, say $x^2 + \beta_1 y + \gamma_1 = 0$, and $x^2 + \beta_2 y + \gamma_2 = 0$, we can eliminate x, *i. e.*, u and v, at one stroke, and get an equation in y alone; from which, by separating reals and imaginaries, we find U and V. Using these, we next find u and v. This is a generalization of the ordinary case.

Conclusion.

Having gone thus far, we are at the threshold of the "theory of functions," which has for its object the study of the transformations effected by the various types of the functional operator "f," on the equation $y = f(x)$. One function depicts the whole x-plane in a sector only of the y-plane—another on a horizontal band or strip—a third, in the interior of a circle—etc. What kind of transformation, for instance, is brought about by the functions log, sin, cosh, exp, cube-root? Or again, given a y-domain with prescribed boundaries, zeros, infinities, singularities—what form of function "f" would produce this effect? In dealing with these questions, graphs of the kind explained above are constantly used.

The chief points of interest to the student in our discussion will be the use of a uniform and fixed principle in graphing $f(x)$—and the manner in which we have

bridged the gap between analytic geometry and function-theory, by way of the fourth dimension. There are many other ways of graphing imaginaries, no doubt; but the above is to be preferred, because it is a logical and systematic scheme resting at both ends on the approved processes.

AN INTRINSIC EQUATION SOLUTION OF A PROBLEM OF EULER.

By PAUL R. RIDER, Washington University.

In his *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, Lausanne (1744), p. 64, Euler proposed and solved the following problem: Given two points P_0 and P_1, and directed lines P_0Q and QP_1 through them, to determine an arc tangent to these two lines at P_0 and P_1, which with its evolute and its normals at P_0 and P_1 will enclose the minimum area. This area is of course expressed by the definite integral $\int_{s_0}^{s_1} \frac{1}{2}\rho ds$, in which ρ is the radius of curvature. The ordinary method of solution is to express the integral in the form

$$\int_{x_0}^{x_1} \frac{(1+y'^2)^2}{2y''}\,dx, \tag{1}$$

and to find, by methods of the calculus of variations, the form of the curves that will give it a minimum value. (It is to be observed that y'', the second derivative of y with respect to x, occurs in the integrand. The problem is of particular interest on this account.) It is necessary to make some substitutions and transformations that are not altogether simple, before it is evident that the minimizing curves are cycloids.[1]

I wish to give here a shorter solution by obtaining the intrinsic equation of the extremals (*i. e.*, minimizing curves). The intrinsic equation of a curve is defined as the relation between s, the length of arc measured from a fixed point on it, and ϕ, the angle of deviation of the tangent at any point from the tangent at the fixed point taken as origin. This is in the sense of Whewell.[2]

If the area integral (1) is written in the form

$$\int_{s_0}^{s_1} \frac{ds}{2\kappa}, \tag{2}$$

where κ is the curvature, we have a special case of an integral studied by Radon,[3]

[1] See, for example, Euler, *loc. cit.*, or the solution outlined by Bolza, *Vorlesungen über Variationsrechnung*, p. 152.

[2] See *Cambridge Philosophical Transactions*, Vols. 8 and 9; or Williamson, *Differential Calculus*, 9th edition, revised (1899), p. 304.

[3] Radon, "Über das Minimum des Integrals $\int_{s_0}^{s_1} F(x, y, \theta, \kappa)ds$," *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften mathematisch-naturwissenschaftliche Klasse*, Wien, Vol. 119 (1910), pp. 1257–1326.

namely,

$$\int_{s_0}^{s_1} F(x, y, \theta, \kappa)ds, \tag{3}$$

where θ is the angle that the tangent line at any point of a curve makes with the x-axis, and κ is the curvature of the curve at the given point. He developed the theory of the calculus of variations for this type of integral, and discussed in detail the case in which F is a function of κ alone. He even mentioned the Euler problem, but did not give a new solution of it, although as will be seen, the intrinsic equation of the extremals can readily and neatly be derived from his results.

For the integral (3) the so-called Euler equations, or differential equations that must be satisfied by minimizing curves, were discovered by Radon to be

$$\begin{aligned} F_x - \frac{d}{ds}\left[\cos\theta(F - \kappa F_\kappa) - \sin\theta\left(F_\theta - \frac{dF_\kappa}{ds}\right)\right] &= 0, \\ F_y - \frac{d}{ds}\left[\sin\theta(F - \kappa F_\kappa) + \cos\theta\left(F_\theta - \frac{dF_\kappa}{ds}\right)\right] &= 0, \end{aligned} \tag{4}$$

subscripts denoting partial differentiation. For the integral (2),

$$F = \frac{1}{2\kappa}, \qquad F_x = F_y = F_\theta = 0, \qquad F_\kappa = -\frac{1}{2\kappa^2},$$

and equations (4) reduce to

$$\begin{aligned} -\frac{d}{ds}\left[\cos\theta\left(\frac{1}{2\kappa} - \kappa\left\{-\frac{1}{2\kappa^2}\right\}\right) - \sin\theta\left(-\frac{d}{ds}\left\{-\frac{1}{2\kappa^2}\right\}\right)\right] &= 0, \\ -\frac{d}{ds}\left[\sin\theta\left(\frac{1}{2\kappa} - \kappa\left\{-\frac{1}{2\kappa^2}\right\}\right) + \cos\theta\left(-\frac{d}{ds}\left\{-\frac{1}{2\kappa^2}\right\}\right)\right] &= 0. \end{aligned}$$

Integration gives

$$\begin{aligned} \cos\theta\cdot\frac{1}{\kappa} - \sin\theta\cdot\frac{d}{ds}\frac{1}{2\kappa^2} &= \alpha, \\ \sin\theta\cdot\frac{1}{\kappa} + \cos\theta\cdot\frac{d}{ds}\frac{1}{2\kappa^2} &= \beta. \end{aligned}$$

Multiplying the first of these equations by $\cos\theta$, the second by $\sin\theta$, and adding, we obtain

$$\frac{1}{\kappa} = \alpha\cos\theta + \beta\sin\theta,$$

or, since $\kappa = \dfrac{d\theta}{ds}$,

$$ds = (\alpha\cos\theta + \beta\sin\theta)d\theta,$$

from which

$$s - s_0 = \alpha\sin\theta - \beta\cos\theta.$$

Let $\alpha = a \cos \theta_0$, $\beta = a \sin \theta_0$. Then

$$s - s_0 = a \sin (\theta - \theta_0).$$

But this is the intrinsic equation of the cycloid.[1]

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

An Elementary Course in Synthetic Projective Geometry. By DERRICK NORMAN LEHMER. Ginn and Company, Boston, 1917. xiii + 123 pages.

Lehmer's *Projective Geometry* "is intended to give in as simple a way as possible the essentials of synthetic projective geometry." The author has done this in the 74 pages of Chapters I—VI, VIII by the methods of pure geometry without the use of anharmonic ratios, circles or any other metric notions. It is his idea that "a purely projective notion ought not to be based on metrical foundations." "The course is not intended to furnish an illustration of how a subject may be developed from the smallest possible number of fundamental assumptions. The author is aware of the importance of work of this sort, but he does not believe it is possible at the present time to write a book along such lines which shall be of much use for elementary students." For the purpose of this course the student should have a thorough knowledge of high-school plane geometry and enough solid geometry to understand the proof of Desargues's theorem about perspective triangles: "*If two triangles ABC and A'B'C' are so situated that the lines AA', BB', CC' all meet in a point, then the pairs of sides AB and A'B', BC and B'C', CA and C'A' all meet on a straight line, and conversely.*"

The first chapter of the book is devoted to the important notion of one-to-one correspondence. In it is explained the need of the fiction about points and lines at infinity. On page 7, after some remarks on one-to-one correspondences being continuous, the author says: "In the case of point-rows this continuity is subject to exception in the neighborhood of the point 'at infinity.'" The reviewer doubts that the student will understand what the "neighborhood of the point at infinity" is.

The theorem of Desargues is very easily proved when the two triangles are not in one plane. When they are in one plane the usual method of proof consists in showing that a third triangle $A''B''C''$ can be found which is perspective to both ABC and $A'B'C'$ from two different centers of projection, and in twice making use of the theorem for two perspective triangles not in one plane. The only difficulty which the student is likely to have with this proof is in the making and the visualizing of the diagram. Lehmer proves the theorem for two triangles not in one plane in the usual way, and draws an illustrative figure;

[1] See Williamson, *loc. cit.*, p. 338.

then he disposes of the more difficult part of the theorem as follows: "If now we consider the figure a plane figure, the points P, Q, and R still all lie on a straight line, which proves the theorem." This appeal to the student's knowledge of perspective drawing really assumes that if ABC and $A'B'C'$ are two perspective triangles in a plane it is always possible to find two other perspective triangles not in one plane of which ABC and $A'B'C'$ are the perspective picture. When this is argued in detail, as it is in the eleventh edition of the "Encyclopedia Britannica," volume 22, page 428, in the article on "Projection," and a diagram is drawn to illustrate, the proof is just as difficult as the usual proof mentioned above. The author should have given a little more space to this important theorem. The proof is hardly convincing in its brief form. In this same chapter are the definitions of four harmonic points, lines and planes and also this definition of projectivity: "Two fundamental forms are projectively related to each other when a one-to-one correspondence exists between the elements of the two and when four harmonic elements of the one correspond to four harmonic elements of the other." One likes to think of the adjective *projective* in the phrase *two projective point-rows* as meaning that one point-row can be obtained from the other by a series of projections and sections; in fact in paragraph 9 in the first chapter our author has used the phrase *projectively related* in just this sense. But the definition of projectivity quoted above has its advantages. Reye uses it in his *Geometrie der Lage;* but later, to use his own words, he "justifies the use of the term *projective*" by proving "that two projective one-dimensional forms may always be considered as the first and last of a series of forms of which each is perspective both to the one preceding it and to the one following it." Lehmer's book does not contain this proof. Reye's proof is based on this fundamental theorem of projective geometry: "*If two projective point-rows, superposed upon the same straight line, have more than two self corresponding points, they must have an infinite number, and every point corresponds to itself; that is, the two point-rows are not essentially distinct.*" Lehmer proves this theorem in Chapter III by making use of the postulate of continuity in this form: "We now assume, explicitly, the fundamental postulate that the correspondence is *continuous*, that is, that *the distance between two points in one point-row may be made arbitrarily small by sufficiently diminishing the distance between the corresponding points in the other.*" In this same Chapter III a point-row of the second order is defined as the locus of the points of intersection of corresponding rays of two projective pencils; and there is a similar definition of a pencil of rays of the second order. Point-rows of the second order are studied in Chapter IV. It is there proved that such a curve is uniquely determined by any five of its points. Pascal's theorem is proved in its original form and also in the cases in which the inscribed hexagon has degenerated into a pentagon, a quadrangle, and a triangle. Enough construction problems are given to show the great power of the theorem. It is pointed out that circles and conic sections are point-rows of the second order and the statement is made that "it will appear later that a point-row of the second order is a conic section. In the future, therefore, we shall refer to the point-row of the second order as a conic."

Chapter V is a similar chapter on pencils of rays of the second order; the development is about the same as in Chapter IV with Brianchon's theorem taking the place of Pascal's. At the end of this chapter the author calls attention to the principle of duality as evidenced by a comparison of Chapters IV and V. In some books the principle of duality is only a working principle; its validity is not rigorously established but the method of translating from a theorem to its correlative is explained and the validity of the latter is established by translating the proof of the former. In such books it is quite the custom to exhibit the theorem and its correlative in parallel columns on the same page. In other books, for example in "Projective Geometry" by Veblen and Young, it is shown that the duality exists in the axioms on which projective geometry is based, and it is argued that the duality therefore exists in the theorems derivable from those axioms. Poncelet, the discoverer of the principle, established its validity by means of the theory of poles and polars. This is the method used by Lehmer in Chapter VI. The starting point of this chapter is the following pair of theorems: "If a quadrangle be inscribed in a conic, two pairs of opposite sides and the tangents at opposite vertices intersect in four points, all of which lie on a straight line." "If a quadrilateral be circumscribed about a conic, the lines joining two opposite points of contact and the lines joining two pairs of opposite vertices are four lines which meet in a point." Lehmer is not always careful to distinguish between quadrangle and quadrilateral. For instance, in the first of these two theorems he says quadrilateral when he means quadrangle.

An involution of points on a line is often defined as the special case of two superposed point-rows in which every point on the line has the same correspondent whether it is thought of as belonging to the one point-row or the other. In Chapter VIII Lehmer introduces the student to the study of involution in a simpler way by first defining, in terms of a quadrangle and a transversal, what is meant by saying that three pairs of points on a line are in involution and by using this definition to show what is meant by saying that all the points of a line are in involution. Later he shows that points in involution on a line form two superposed projective point-rows.

The seven chapters of 74 pages that have been mentioned constitute Lehmer's course in the essentials of synthetic projective geometry. But this is not all that the book contains. Chapter VII is entitled "Metrical Properties of the Conic Sections." It is here that the author proves that point-rows of the second order are really conic sections. He gets hold of the metrical properties by means of the infinitely distant elements of the plane. The polar line of an infinitely distant point is a diameter, the pole of the infinitely distant line is the center, etc. Point-rows of the second order are classified as hyperbola, parabola and ellipse according as the curve has two points, one point or no point at infinity; and then their analytic geometry equations are found to be the well-known forms $xy = constant$, $y^2 = 2px$, *and* $x^2/a^2 + y^2/b^2 = 1$. This identifies the point-rows of the second order with the hyperbola, parabola and ellipse as they are known to the student of analytic geometry. If that student had in his analytic

geometry course the proof that these curves are really sections of a circular cone, he is then sure that point-rows of the second order are really conic sections. The reviewer has a preference for another method of proving that point-rows of the second order are the same as conic sections, namely the method given in Cremona's *Projective Geometry*, where it is proved that the locus of the points of intersection of corresponding rays of two projective point-rows is a figure in homology with a circle. Of course this necessitates a chapter on homology which Lehmer's book does not have. If synthetic projective geometry is to be studied by students who have not had analytic geometry, and Lehmer states in the preface his belief that it is "destined shortly to force its way down into the secondary school," it seems to the reviewer that the identifying of point-rows of the second order with conic sections had better be done by means of homology. The book as it stands has too many references to analytic geometry to be suitable for the student who has not had a course in that subject. And if the references to analytic geometry are omitted, the course seems incomplete.

Chapter IX is entitled "Metrical Properties of Involutions." In it are defined axes, foci and directrices of conics, and the well-known focus-directrix property is established for all the conics.

Chapter X, consisting of 23 pages, gives a good account of the historical development of the subject.

For reasons already mentioned it seems to the reviewer that the book is not well adapted for use by those who have not studied analytic geometry. But it can be made the basis of a good semester course for college juniors and seniors if supplemented by lectures on problems of the second degree and perhaps other topics. As it stands it is hardly extensive enough.

One other comment comes to mind. The principle of duality is responsible for much awkward phraseology. Students and writers of textbooks too often write the correlative of a statement by merely putting the word plane for point and point for plane and thereby getting a sentence which is at best an awkward statement of what was intended. The one such example in this book which caught the reviewer's attention is this sentence in paragraph 16 of Chapter I. "The point, considered as made up of all the lines and planes passing through it, is called a point-system." To speak of a point as made up of lines and planes will strike the student as absurd. Mathematical literature has altogether too many such statements that strike the reader as absurd, although the absurdity may be minimized when the reader understands the origin of the phraseology. Lehmer gives an interesting comment along this line in paragraph 181 in the chapter on the history of synthetic geometry: "When the geometer speaks of the two real or imaginary intersections of a straight line with a conic, he is really speaking the language of algebra. *Apart from the algebra involved*, it is the height of absurdity to try to distinguish between the two points in which a line *fails to meet a conic.*

W. H. BUSSEY.

THE UNIVERSITY OF MINNESOTA.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

489. Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove the following:

$$\begin{vmatrix} -x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ -v & -u & y & -x \end{vmatrix}^2 + a \begin{vmatrix} x & -x & -bu & abv \\ y & y & -bv & -bu \\ u & u & x & ay \\ v & -v & y & -x \end{vmatrix}^2$$

$$+ b \begin{vmatrix} x & -ay & -x & abv \\ y & x & y & -bu \\ u & av & u & ay \\ v & -u & -v & -x \end{vmatrix}^2 + ab \begin{vmatrix} x & -ay & -bu & -x \\ y & x & -bv & y \\ u & av & x & u \\ v & -u & y & -v \end{vmatrix}^2 = \begin{vmatrix} x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ v & -u & y & -x \end{vmatrix}^2.$$

490. Proposed by HENRY HEATON, Atlantic, Iowa.

Show that $\sin 3^\circ = \frac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \frac{1}{8}(\sqrt{5 + \sqrt{5}} - \sqrt{15 + 3\sqrt{5}})$.

491. Proposed by J. W. LASLEY, University of North Carolina.

Solve the equations $xy = x^2 - y^2$ and $x^2 + y^2 = x^3 - y^3$ for x and y.

GEOMETRY.

522. Proposed by GEORGE Y. SOSNOW, Newark, N. J.

Prove that the sum of the squares of the edges of a tetrahedron is equal to four times the sum of the squares of the lines joining the middle points of the opposite edges.

523. Proposed by H. CAMPBELL, St. Johnsbury, Vt.

Given the difference of the segments of the base made by the perpendiculars let fall from the vertical angle, the difference of the base angles, and the sum of the sides of a triangle, to construct the triangle.

CALCULUS.

437. Proposed by LEIGH PAGE, Yale University.

Integrate

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx$$

without the use of Gamma Functions.

438. Proposed by PAUL CAPRON, U. S. Naval Academy.

Find the locus of the equation

$$y^6 - 3(a^2 - x^2)y^4 - 2a^2x^2y^3 + 3(a^2 - x^2)y^2 - 6ax^2(a^2 - x^2)y + a^2x^4 - (a^2 - x^2)^3 = 0,$$

first showing that it can be reduced to the form $y = kx^n \pm (a^2 - x^2)^m$, and finding the points of maximum abscissas, of maximum ordinates, and of inflection.

439. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

MECHANICS.

354. Proposed by G. PAASWELL, New York City.

The acceleration of an electric train is constant and equal to a feet per sec. per sec. Its braking or deceleration is variable and equal to the square root of the velocity. If the distance between stations is 5,000 feet, show that the acceleration must cease and braking ensue when the train is about 960 feet from the stopping point; also that the maximum velocity attained for a minimum time run is 88 m.p.h. and the time of run 54 seconds.

355. Proposed by HORACE OLSON, Chicago, Illinois.

A solid spheroid, axes a, a, b, is placed with its axis of revolution vertical. From its highest point, a particle is projected horizontally with a speed s. Where will it leave the spheroid, assuming that it slides on the surface without friction?

NUMBER THEORY.

272. Proposed by C. C. YEN, Tangshan, North China.

How many integers prime to n are there in each of the sets:

(a) $1\cdot 2,\ 2\cdot 3,\ 3\cdot 4,\ \cdots,\ n(n+1)$;

(b) $1\cdot 2\cdot 3,\ 2\cdot 3\cdot 4,\ 3\cdot 4\cdot 5,\ \cdots,\ n(n+1)(n+2)$;

(c) $\frac{1\cdot 2}{2},\ \frac{2\cdot 3}{2},\ \frac{3\cdot 4}{2},\ \cdots,\ \frac{n(n+1)}{2}$;

(d) $\frac{1\cdot 2\cdot 3}{6},\ \frac{2\cdot 3\cdot 4}{6},\ \frac{3\cdot 4\cdot 5}{6},\ \cdots,\ \frac{n(n+1)(n+2)}{6}$?

273. Proposed by V. M. SPUNAR, Chicago, Illinois.

The ratio of the chances that all numbers ending in 1 or 9 and those ending 3 or 7 are composite is 3 : 2.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

477. Proposed by J. L. RILEY, Junior College, Stephenville, Texas.

Evaluate the product $(1+r+r^2+r^3)(1+r^3+r^4+r^6)\cdots(1+r^{2^{n-1}}+r^{2^n}+r^{3\cdot 2^{n-1}})$.

SOLUTION BY LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Since

$$1+r+r^2+r^3=\frac{1-r^4}{1-r},\quad 1+r^2+r^4+r^6=\frac{1-r^8}{1-r^2},\quad 1+r^4+r^8+r^{12}=\frac{1-r^{16}}{1-r^4},\quad \cdots,$$

$$1+r^{2^{n-1}}+r^{2^n}+r^{3\cdot 2^{n-1}}=\frac{1-r^{2^{n+1}}}{1-r^{2^{n-1}}},$$

the required product is equal to the product of the following fractions:

$$\frac{1-r^4}{1-r}\cdot\frac{1-r^8}{1-r^2}\cdot\frac{1-r^{16}}{1-r^4}\cdot\frac{1-r^{32}}{1-r^8}\cdots\frac{1-r^{2^{n-1}}}{1-r^{2^{n-2}}}\cdot\frac{1-r^{2^n}}{1-r^{2^{n-2}}}\cdot\frac{1-r^{2^{n+1}}}{1-r^{2^{n-1}}}.$$

The result, manifestly, is

$$\frac{(1-r^{2^n})(1-r^{2^{n+1}})}{(1-r)(1-r^3)}.$$

Also solved by H. C. FEEMSTER, PAUL CAPRON, ELIJAH SWIFT, and HORACE OLSON.

478. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Solve the equations, $l/x + y/m + z/n = 1$, $x/l + m/y + z/n = 1$, $x/l + y/m + n/z = 1$.

SOLUTION BY HENRY D. THOMPSON, Princeton N. J.

Take X, Y, Z as symbols for $x/l, y/m, z/n$, respectively. Then the equations to be solved are:

$$(1)\ 1/X + Y + Z = 1, \quad (2)\ X + 1/Y + Z = 1, \quad (3)\ X + Y + 1/Z = 1.$$

The difference of (2) and (1) gives (4) $(X - Y) + (X - Y)/XY = 0$, one solution of which is (5) $X = Y$. This substituted in (1), (2), (3) leaves to be solved (6) $1/X + X + Z = 1$, (7) $2X + 1/Z = 1$, and (7) gives $Z = 1/(1 - 2X)$.

This in (6) gives the cubic

$$1/X + 1/(1 - 2X) = 1 - X, \quad \text{or} \quad (1 - X)/X(1 - 2X) = (1 - X),$$

the solutions of which are $X = 1$, and $X = 1/4 \pm \sqrt{-7}/4$. These with (5) and (7) give for $[x, y, z]$ the three solutions:

$$(\text{I, II})\ [(1 \pm \sqrt{-7})l/4,\ (1 \pm \sqrt{-7})m/4,\ (1 \pm \sqrt{-7})n/4], \quad (\text{III})\ [l, m, -n].$$

The other solution of (4) viz.: (8) $1 + 1/XY = 0$ in (1) or (2) gives $Z = 1$, whence (3) becomes $X + Y = 0$, which with (8) gives $X^2 - 1 = 0$, and $X = 1$ with $Y = -1$, or $X = -1$ with $Y = 1$. Two other solutions are:

$$(\text{IV})\ [l, -m, n], \quad (\text{V})\ [-l, m, n].$$

To find the infinite solutions, set $X/W, Y/W, Z/W$, respectively, for X, Y, Z in (1), (2), (3) and these equations rid of denominators, and with $W = 0$ are:

$$(9)\ X(Y + Z) = 0, \quad (10)\ Y(Z + X) = 0, \quad (11)\ Z(X + Y) = 0.$$

The equation (9) is satisfied when $X = 0$, which set in (10) or (11) gives either $Y = 0$ or $Z = 0$. The other solution of (9) viz.: (12) $Y = -Z$ set in (10) and (11) gives $Y(X - Y) = 0$ and $-Y(X + Y) = 0$; whence $Y = 0$, and from (12), $Z = 0$. Thus the infinite solutions (VI), (VII), (VIII) are the points at infinity on the axes when x, y, z are taken as coördinates in a rectilineal system.

Note that each of the three hyperbolic cylinders (1), (2), (3) has for its asymptotic planes a coördinate plane and a plane parallel to the like named axis.

Also solved by B. J. DINING, GERTRUDE I. MCCAIN, J. L. RILEY, E. F. CANADY, S. W. REAVES, J. Q. MCNATT, G. W. HARTWELL, R. A. JOHNSON, C. H. WORTHINGTON, PAUL CAPRON, HORACE OLSON, and G. Y. SOSNOW.

479. Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove

$$\left\{\begin{vmatrix} x & -v & -z \\ -y & -z & v \\ -z & y & -x \end{vmatrix}^2 + \begin{vmatrix} y & -v & -z \\ x & -z & v \\ v & y & x \end{vmatrix}^2 + \begin{vmatrix} x & y & -z \\ -y & x & v \\ -z & v & -x \end{vmatrix}^2 + \begin{vmatrix} x & -v & y \\ -y & -z & x \\ -z & y & v \end{vmatrix}^2 \right\}$$

$$\div \begin{vmatrix} x & -y & -z & v \\ y & x & -v & -z \\ z & v & x & y \\ v & -z & y & -x \end{vmatrix}^2 = (x^2 + y^2 + z^2 + v^2)^{-1}.$$

SOLUTION BY HENRY D. THOMPSON, Princeton, N. J.

Let Q be a symbol for $(x^2 + y^2 + z^2 + v^2)$, and represent the determinants in the order in which they appear in the equation by Z, V, X, Y, H, so that the equation is

$$\{Z^2 + V^2 + X^2 + Y^2\} \div H^2 = Q^{-1}.$$

Application of the rule for the multiplication of determinants gives:

$$Z^2 = \begin{vmatrix} Q - y^2 & -yx & -yv \\ -xy & Q - x^2 & -xv \\ -vy & -vx & Q - v^2 \end{vmatrix}.$$

Dividing the first, second, and third *columns* by y, x, v, respectively, and *multiplying* the first, second, and third *rows*, respectively, by the same symbols gives:

$$Z^2 = \begin{vmatrix} Q - y^2 & -y^2 & -y^2 \\ -x^2 & Q - x^2 & -x^2 \\ -v^2 & -v^2 & Q - v^2 \end{vmatrix},$$

which easily reduces to $Z^2 = Q^2(Q - v^2 - y^2 - x^2) = Q^2z^2$.

In exactly the same way, $V^2 = Q^2v^2$, $X^2 = Q^2x^2$, $Y^2 = Q^2y^2$. Whence

$$\{Z^2 + V^2 + X^2 + Y^2\} = Q^2(z^2 + v^2 + x^2 + y^2) = Q^2.$$

Direct application of the rule for the multiplication of determinants gives for H^2 a determinant wherein each element in the leading diagonal is Q, and all the other elements are zero; whence $H^2 = Q^4$. These values set in the equation prove it to be correct.

Also solved by J. L. RILEY, E. H. WORTHINGTON, A. M. HARDING, J. B. ROSENBACH, and the PROPOSER.

GEOMETRY.

511. Proposed by FRANK V. MORLEY, Student, Haverford, College.

Let a_i $(i = 1, 2, 3, 4)$ be four points on a circle, and let the incenter of the triangle formed by omitting a_i be c_i; prove that the four points c_i form a rectangle.

SOLUTION BY THE PROPOSER.

Let there be four circles with centers, m_{12}, m_{23}, m_{34}, m_{41}, on the given circle, and let each circle intersect the next in two points, of which one set, a_2, a_3, a_4, a_1, lie on the given circle, and one set, c_4, c_1, c_2, c_3, lie inside, as shown in Fig. 1.

Then $\measuredangle c_4a_1a_2 = \frac{1}{2}\measuredangle c_4m_{12}a_2$, and $\measuredangle m_{23}a_1a_2 = \frac{1}{2}a_3m_{12}a_2$ (1).

$\therefore \measuredangle c_4c_3a_2 = \measuredangle c_4a_1a_2 = \measuredangle m_{23}a_1a_2 = \measuredangle m_{23}m_{41}a_2$, and $c_3c_4 \parallel m_{41}m_{23}$.

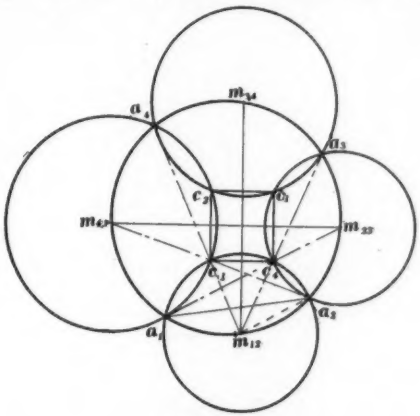


FIG. 1.

By equality of arcs, $m_{41}m_{23} \perp m_{12}m_{34}$, and hence $c_3c_4 \perp m_{12}m_{34}$.

Since $\measuredangle a_4m_{12}a_3 \equiv \measuredangle c_3m_{12}c_4$, and is bisected by $m_{12}m_{34}$, c_3, c_4 are reflections through $m_{12}m_{34}$, and likewise c_1, c_2; similarly c_1, c_4, and c_2, c_3, are reflections through $m_{41}m_{23}$.

$\therefore$ c_1, c_2, c_3, c_4, form a rectangle.

But by (1), c_4 is the intersection of a_3m_{12}, a_1m_{23}, or the incenter of $a_1a_2a_3$. Therefore c_i as defined above is identical with the incenter of the triangle formed by omitting a_i, and the problem is proved.

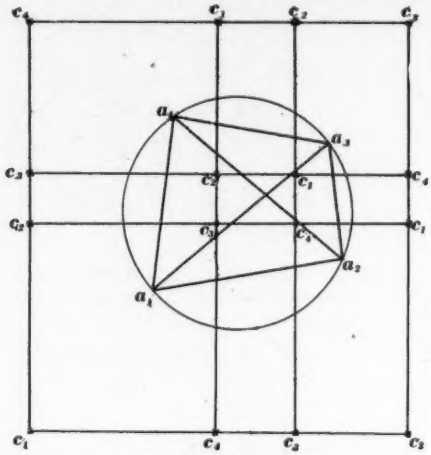


FIG. 2.

The extension suggested to be proved by analogous methods is shown in Fig. 2, where the four incenters and twelve excenters, of the four triangles formed by omitting a_i in turn, are the intersections of two sets of four perpendicular lines.

Also solved by O. J. RAMLER, R. A. JOHNSON, J. E. ROWE, J. W. CLAWSON, and PAUL CAPRON.

CALCULUS.

409. Proposed by B. J. BROWN, Victor, Colorado.

Integrate the differential equation,

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - \frac{2}{(x+y)^2} z = 0.$$

II. SOLUTION BY W. W. BEMAN, Ann Arbor, Michigan.

This problem is found in Gregory's *Examples* and Forsyth's *Treatise on Differential Equations*. Gregory puts

$$y + x = u, \qquad y - x = v$$

and makes use of the solution of a previous problem.

Maser in the German translation of Forsyth's *Treatise* puts

$$z = \frac{u}{(x+y)^2} \text{ and writes the new equation } (x+y)\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.$$

Differentiating with respect to x, he gets

$$(x+y)\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^2 u}{\partial x^2}, \text{ and from this } \frac{\partial^2 u}{\partial x^2} = (x+y)\phi'''(x).$$

Integrating twice with respect to x, he obtains u and finally z as in Mr. Adams's solution on p. 129 of the March, 1917, MONTHLY. It may be observed that in Mr. Adams's solution

$$w = \frac{\partial^2 u}{\partial x^2}.$$

418. Proposed by B. F. FINKEL, Drury College.

A rectangular tract of land is to be bought for the purpose of laying out a quarter-mile track with straightaway sides and semicircular ends. In addition a strip 35 yards wide along each straightaway is to be bought for grandstands, training quarters, etc. If the land costs \$200 an acre, what will be the least possible cost of the land required?

Granville's *Differential and Integral Calculus*, p. 116.

Is there anything wrong with this problem? Explain the contradiction involved in the solution.

SOLUTION BY CLARIBEL KENDALL, University of Colorado.

For the cost of the land in this problem to be a minimum the area must be a minimum.

Let x = radius of ends in yards, y = length of straightaway sides in yards, and v = value of land.

Then $(2x + y)2x$ = area of the rectangular tract, and $70y$ = area of grandstands.

(1) $A = 70y + (2x + y)2x$, total area to be bought.

(2) $2\pi x + 2y = 440$ yds., length of the track, or $y = 220 - \pi x$, $y' = -\pi$, where $y' = dy/dx$.

(3) $A' = 70y' + 8x + 2xy' + 2y = 0$; or, substituting for y and y',

$$(4 - 2\pi)x = 35\pi - 220, \quad \text{and} \quad x = \frac{220 - 35\pi}{2\pi - 4} = \frac{385}{8} \text{ yds.} \left(\pi = \frac{22}{7}\right).$$

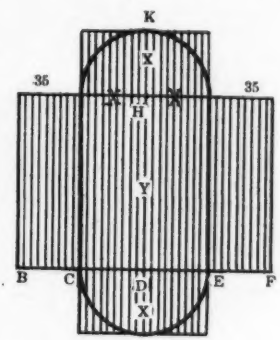


FIG. 1.

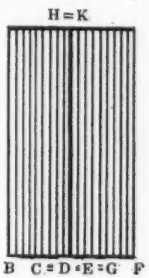


FIG. 2.

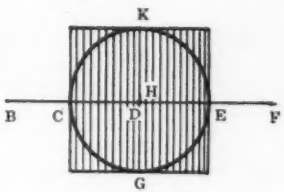


FIG. 3.

Whence

$$y = \frac{275}{4} \text{ yds.}, \quad \text{and} \quad A = 70y + 2x(2x + y) = \frac{824775}{4} \text{ sq. yds.} = 4.28 \text{ acres.}$$

At \$200 an acre, the cost v is \$856 as given by Granville.

From (3) above

$$A'' = 8 - 4\pi,$$

showing that A'' is a negative constant. This proves that A is a maximum, and that no minimum is even possible, under the customary definitions of maxima and minima.

We next inquire whether there may not be, after all, some sort of practical minimum, based on the exclusion of negative values for the end-radius x and for the straightaway side y.

From (2) above $\pi x + y = 220$. Plainly, x must lie between 0 and 70, while y must lie between 220 and 0.

When $x = 0$, $y = 220$; $A = 15400$, $v = \$636.36$. The land bought is now a rectangle. The semi-circular ends have degenerated into points and the track is a strip of zero width between the front edges of the grandstands.

When $x = 70$, $y = 0$; $A = 19600$, $v = \$809.92$, and the ground bought is now a square, the track is the inscribed circle, while the stands reduce to the 35-foot extensions of one of the two diameters of contact.

As x increases from 0 to 70, A and v increase up to the maximum and then decrease. The values for A and v are greater, however, for $x = 70$ than for $x = 0$. (See Fig. 4.) A practical

minimum for v may therefore be secured by taking x as small as possible to allow for an easy-running curve at the ends.

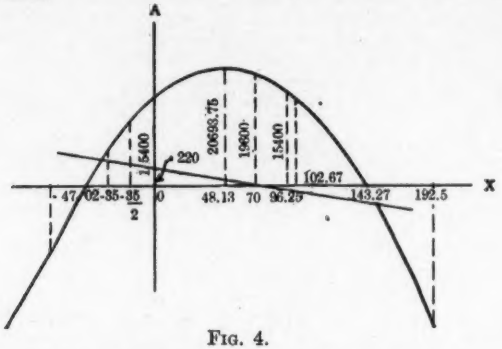


FIG. 4.

As far as has been discovered, there is no contradiction involved in the solution, but rather a misconception in the statement of the problem.

Aside from the question of determining a maximum or a minimum, this problem is curious when we try to represent the area for negative values of x and y. We will trace x from $-\infty$ to $+\infty$.

The following conventions will be adopted: Positive areas will be designated by vertical shadings; negative areas by horizontal shadings, positive areas counted twice by diagonal cross-bar shadings; positive and negative areas cancelling one another by vertical and horizontal shadings (negative areas counted twice do not occur). The positive part of the race track will be designated by a heavy solid line, the negative part by a heavy dotted line. In the following figures, as in Figs. 1, 2, 3, $BC(=EF=35)$ the width of the grandstands will be considered as essentially positive and these grandstands always face in opposite directions; $x = CD = DE$, with the usual conventions as to directed line segments, positive when measured to the right and negative when measured to the left; $y = DH$, positive when measured upward, negative when measured downward; $70 + 2x = BF$, $2x + y = GK$, with the same sign-conventions as above. One of the semi-circular ends of the track always goes from C to E and correspondingly for the other end.

The area of the tract of land may be represented in two ways:

$$A = (70 + 2x)y + 4x^2, \tag{1a}$$

$$A = 70y + (2x + y)2x. \tag{1b}$$

$4x^2$ represents the area of the two rectangles, including the semi-circular ends of the race-track; $(70 + 2x)y$ is the area of the rectangle having BF and DH as dimensions; $70y$ is the area of the two grandstands; $(2x + y)2x$ is the area of the rectangle having CE and GK as dimensions.

For $-\infty < x < 0$, $y > 0$ it is preferable to consider A in the form (1a). $4x^2$ is always positive and since $y > 0$,

$$(70 + 2x)y \gtreqless 0 \text{ according as } 70 + 2x \gtreqless 0, \text{ } i.\ e., \text{ for } x \gtreqless -35. \tag{4}$$

For $0 \leqq x \leqq 70$, $y \geqq 0$ either form is applicable since all the above areas are then positive or zero.

For $70 < x < +\infty$, $y < 0$ the form (1b) is preferable. $70y$ is always negative, and since $x > 0$,

$$(2x + y)2x \gtreqless 0 \text{ according as } 2x + y \gtreqless 0, \text{ } i.\ e., \text{ for } y \gtreqless -2x(=-385) \text{ or } x \gtreqless 192\tfrac{1}{2}(=192.5), \tag{5}$$

obtained by substituting for y in equation (2).

In the following constant reference should be made to the graph in Fig. 4.

I. When $x < -47.02$ then $y > 0$ and $A < 0$. The tract of land is represented by Fig. 5, $(70 + 2x)y < 0$ by (4) since $x < -35$. And since $A < 0$, $|(70 + 2x)y| > 4x^2$, *i. e.*, the area of the rectangle $BF \times DH$ is greater than the area of the rectangles including the semi-circular ends of the race track.

II. When $x = -47.02$ then $y > 0$ and $A = 0$. Fig. 5 is again representative, since $x < -35$ but now $|(70 + 2x)y| = 4x^2$, *i. e.*, the positive and negative areas just balance.

III. When $-47.02 < x < 0$ then $y > 0$ and $A > 0$. This must be divided into three special cases dependent upon the three parts of (4).

(*a*) When $-47.02 < x < -35$, $(70 + 2x)y < 0$ by (4) but now $|(70 + 2x)y| < 4x^2$ since $A > 0$, *i. e.*, the positive part of the area predominates. Fig. 5 is again representative.

(*b*) When $x = -35$, $(70 + 2x)y = 0$ by (4) and $A = 4x^2$, a positive quantity. Fig. 6 represents this condition.

(*c*) When $-35 < x < 0$, $(70 + 2x)y > 0$ by (4). All of the rectangular areas are now positive. Figures 7, 8, 9 represent the tract for different values of x, as shown under the cuts.

In Fig. 8, the grandstands occupy the same base but face in opposite directions.

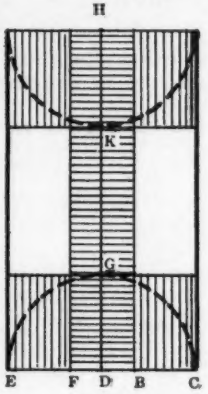


FIG. 5.
$-\infty < x < -35$.

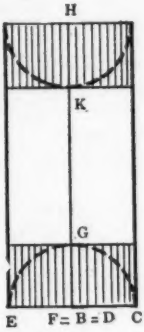


FIG. 6.
$x = -35$.

FIG. 7.
$-35 < x < -\frac{35}{2}$.

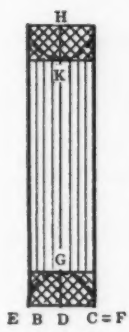


FIG. 8.
$x = -\frac{35}{2}$.

FIG. 9.
$-\frac{35}{2} < x < 0$.

In I, II, III the race-track is partly positive and partly negative. In equation (2), $440 = 2y + 2\pi x$, the $2y$ is positive and the $2\pi x$ is negative, as indicated in the figures by the solid and dotted parts of the track. Whether the negative part might mean that the runner is running backwards has not been investigated.

IV. When $0 \leqq x \leqq 70$ then $y \geqq 0$ and $A > 0$. These cases have been discussed in the solution of the original problem. As stated above either form of (1) may be used.

In IV the entire race-track is positive.

V. When $70 < x < 143.27$ then $y < 0$ and $A > 0$. Since $x < 192.5$, $(2x + y)2x > 0$ by (5). And since $A > 0$, $(2x + y)2x > |70y|$, *i. e.*, the area of the rectangle $GK \times CE$ is greater than the area of the grandstands. This divides into several cases for

(6) $\qquad y \gtrless -x$, *i. e.*, $x \gtrless \frac{210}{2}(= 102.67)$ (obtained by substituting for y in (2))

since $x = 102.67$ lies in the interval under discussion. Figures 10, 11, 12 are representative for different values of x, as shown under the cuts.

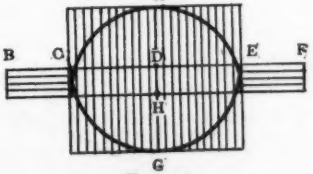


FIG. 10.
$70 < x < 102.67$, $y > -x$.

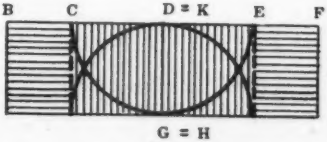


FIG. 11.
$x = 102.67$, $y = -x$.

VI. When $x = 143.27$ then $y < 0$ and $A = 0$. Since $x < 192.5$, $(2x + y)2x > 0$ by (5). And since $A = 0$, $(2x + y)2x = |70y|$. Fig. 12 is again representative and the positive and negative areas just balance.

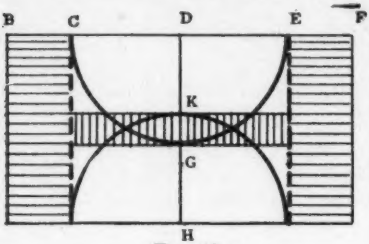


FIG. 12.
$102.67 < x < 192.5$, $y < -x$.

VII. When $x > 143.27$ then $y < 0$ and $A < 0$. On account of (5) we must consider three cases.

(*a*) When $143.27 < x < 192.5$, $(2x + y)2x > 0$ by (5). And since $A < 0$, $(2x + y)2x < |70y|$, *i. e.*, the area of the rectangle $GK \times CE$ is less than the area of the grandstands. Fig. 12 is again representative.

(*b*) When $x = 192.5$, $(2x + y)2x = 0$ by (5) and $A = 70y$, a negative area. Fig. 13 represents the tract.

(*c*) When $x > 192.5$, $(2x + y)2x < 0$ by (5). Now all of the areas included in A are negative and Fig. 14 is representative.

In V, VI, VII, the race-track is partly positive and partly negative. In equation (2) the $2\pi x$ is positive and the $2y$ is negative as is indicated in the figures by the solid and dotted parts of the track.

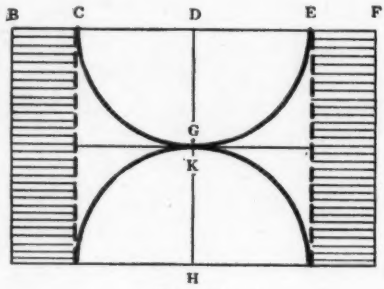


FIG. 13.
$x = 192.5$, $y = -2x$.

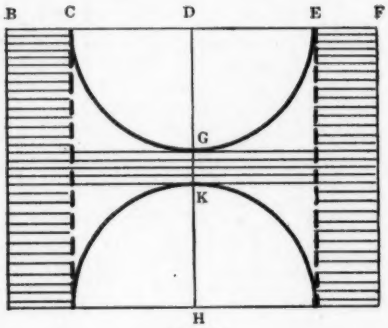


FIG. 14.
$192.5 < x + \infty$, $y < -2x$.

Also solved by HORACE OLSON, S. E. URNER, PAUL CAPRON, and J. W. BALDWIN.

MECHANICS.

336. Proposed by C. N. SCHMALL, New York City.

An inclined plane, length l, makes an angle $\phi < (\frac{1}{4}\pi)$ with the horizontal plane through its foot. From its foot, a body is projected upward along the plane, with a velocity equal to that of a falling body at the height h, so as to pass over the top and strike the horizontal plane at the maximum distance, x, from the foot of the inclined plane. Show by methods of the calculus that $x = h/(\sin\phi \cos\phi)$, and that the corresponding value of l is $(2h \cot 2\phi)/\cos\phi$.

SOLUTION BY THE PROPOSER.

Let v_1 = velocity at time of projection; v = velocity on reaching top of plane; a = height of inclined plane; x_1 = horizontal distance traversed by body after leaving plane.

Then, by the given conditions, we have

$$v_1^2 = 2gh, \tag{1}$$

$$a = l \sin \phi, \tag{2}$$

$$v^2 = 2g(h - a), \tag{3}$$

$$x = x_1 + a \cot \phi, \tag{4}$$

or,

$$x_1 = x - a \cot \phi. \tag{5}$$

Now taking the top of the inclined plane as the origin, the equation of the path of the body is

$$\begin{aligned} -a &= x_1 \tan \phi - \frac{g}{2} \frac{x_1^2}{v^2 \cos^2 \phi} \\ &= x_1 \tan \phi - \frac{g}{2} \frac{x_1^2}{2g(h-a) \cos^2 \phi} \end{aligned} \tag{6}$$

by equation (3). Whence,

$$(h - a)(a + x_1 \tan \phi) = \frac{x_1^2}{4 \cos^2 \phi}$$

which, putting $x - a \cot \phi$ for x_1 from (5), becomes

$$(h - a)x \tan \phi = \frac{(x - a \cot \phi)^2}{4 \cos^2 \phi}.$$

Hence,

$$x^2 - 2x(a \cot \phi \cos 2\phi + h \sin 2\phi) + a^2 \cot^2 \phi = 0, \tag{7}$$

where x is to be made a maximum.

Differentiating with respect to a, we get

$$\{x - (a \cot \phi \cos 2\phi + h \sin 2\phi)\} \frac{dx}{da} - x \cot \phi \cos 2\phi + a \cot^2 \phi = 0. \tag{8}$$

Hence,

$$\frac{dx}{da} = 0, \qquad \text{if} \qquad x = a \frac{\cot \phi}{\cos 2\phi}.$$

Putting this value of x in (7) and reducing, we get

$$a = h \frac{\cos 2\phi}{\cos^2 \phi}. \tag{9}$$

Again, substituting this value of x in the coefficient of dx/da in (8), we get

$$a \frac{\cot \phi}{\cos 2\phi} - a \cot \phi \cos 2\phi - h \sin 2\phi,$$

which is nearly equal to $h \sin 2\phi$.

It is evident, therefore, that as a increases through the value $h(\cos 2\phi)/(\cos^2 \phi)$, the coefficient of dx/da remains about equal to $h \sin 2\phi$, a positive quantity, while the remainder of equation (8), namely, $-x \cot \phi \cos 2\phi + a \cot^2 \phi$, changes from $-$ to $+$, for x is constant and a is increasing. Hence, dx/da changes from $+$ to $-$. Hence x is a maximum when

$$x = a \frac{\cot \phi}{\cos 2\phi} = \frac{h}{\sin \phi \cos \phi}. \qquad \text{by (9)}$$

Also, from (2), we get

$$l = \frac{a}{\sin \phi} = 2h \frac{\cot 2\phi}{\cos \phi}. \qquad \text{by (9)}$$

Also solved by O. S. ADAMS and HORACE OLSON.

NUMBER THEORY.

255. Proposed by FRANK IRWIN, University of California.

Given any arithmetical progression whose first term a and common difference d are relatively prime integers, and any finite set of positive integers $m_1, m_2, \cdots$ also relatively prime to d, it is required to determine an integer n such that the multiples of $m_1, m_2, \cdots$ may occupy the same

positions in the series of natural numbers beginning with n as they do in the arithmetical progression. This is to say that if the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th, $\cdots$ terms of the progression are divisible by m_1, so also will be the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th $\cdots$ terms of the series n, $n + 1$, $n + 2$, $\cdots$, etc. Show that n may be determined as the solution of a congruence $An + B \equiv 0 \pmod{C}$ whose coefficients, A, B, are constants independent of the number and value of the m's.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

By the conditions of the problem the following equations hold: (1) $a + (k - 1)d = m_1\alpha$, (2) $n + k - 1 = m_1\beta$, where α and β are integers. Eliminating k, we obtain,

$$dn - a = m_1(\beta d - \alpha). \tag{3}$$

Consequently we see that n must be a solution of the congruence $dn - a \equiv 0 \pmod{\text{L.C.M. of } m_1, m_2, \cdots}$. Conversely any solution of this congruence satisfies the given condition. For let n be such a solution. Then (3) is true. We are to show that if (1) holds, (2) will also: and conversely. But adding the equations (1) and (3) we see that $d(n + k - 1)$ is divisible by m_1, and since d is prime to m_1, $n + k - 1$ is divisible by m_1. Similarly (2) implies (1).

Also solved by C. F. GUMMER and HORACE OLSON.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSION.

RELATING TO A CONSTRUCTION FOR THE GRAPH OF A CUBIC.

BY PAUL CAPRON, U. S. Naval Academy.

If a cubic is given by the equation

$$\alpha y = x^3 + \beta x^2 + \gamma x + \delta, \qquad (\alpha > 0)$$

and the origin is shifted to (x_0, y_0), where

$$x_0 = -\frac{\beta}{3}, \qquad y_0 = \frac{1}{27\alpha}(2\beta^3 - 9\beta\gamma + 27\delta),$$

the new equation is

$$\alpha y = x^3 + (\gamma - \beta^2/3)x.$$

It is convenient to distinguish three cases, according to the value of $(\gamma - \beta^2/3)$.

Case (1). $\gamma - \beta^2/3 < 0$. Write $\alpha = a^2$, $\gamma - \beta^2/3 = -c^2$; then the equation is

$$a^2 y = x^3 - c^2 x.$$

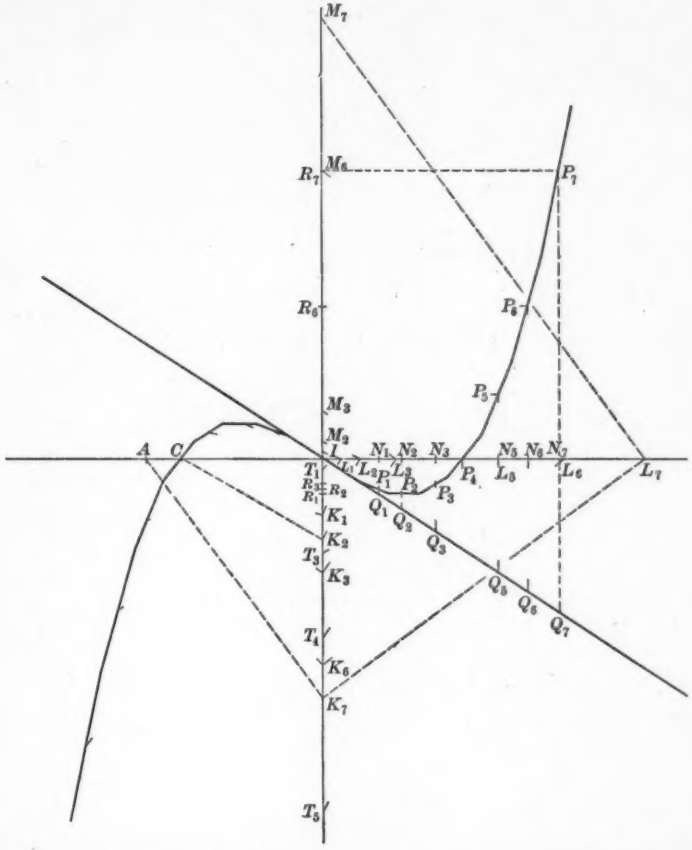
$dy/dx = 1/a^2(3x^2 - c^2)$. At the inflection, I, (0, 0), we have
$dy/dx = -c^2/a^2 = 1/3\alpha(3\gamma - \beta^2)$, and $dy/dx = 0$ at $[\pm c/\sqrt{3}, \mp (2/3\sqrt{3})(c^3/a^2)]$.

At $(\pm c, 0)$, $dy/dx = 2(c^2/a^2)$, numerically twice the slope at I.

$x = a^2/k$, $y = (a^4/k^3) - (c^2/k)$ are parametric equations for the graph.

Hence the following construction for the graph in Case (1):

Locate the inflection I; through it draw AIL and KIM parallel to the axes of abscissas and ordinates respectively, and in the positive direction in each case. Through I draw the inflectional tangent IQ, giving it the slope $-(c^2/a^2) = (1/3\alpha)(3\gamma - \beta^2)$. Make $AI = a$. From A draw AK to any point K of KM; thence $KL \perp AK$; thence $LM \perp KL$. On IL lay off $IN = KI$ (by rotating IK contraclockwise through $\pi/2$); through N draw $PQ \perp IL$. On MI lay off $MR = NQ$ (and in the same direction); through R draw $RP \perp IM$, thus determining a point P. [This construction is made in the figure for the series of points with subscript 7, and the points are indicated (without actually drawing the lines) for those having the subscripts 1, 2, 3 and 6.]



The point P is a point of the graph. The proof rests upon the theorem that if a perpendicular p is drawn from the vertex of a right triangle to the hypotenuse, dividing the hypotenuse into segments m and n, then $mn = p^2$. For if AK' (not shown in the figure) is drawn perpendicular to AK to meet IM at K', and

if IK' (not shown) $= k$, then $KI = a^2/k$; similarly $IL = KI^2/a = a^3/k^2$; $IM = IL^2/KI = a^4/k^3$. Also $IN = KI = a^2/k = x$; $MR = NQ = -(c^2/a^2) \times IN = -(c^2/k)$, and $IR = IM + MR = IM + NQ = (a^4/k^3) - (c^2/k) = y$.

Special Points. The minimum point (P_2 in the figure) can be determined geometrically, and there are two points, P_4 and P_5, for which the complete construction is not necessary.

If we make $CI = c$, and $\measuredangle ICK_2 = 30°$, the point obtained by the construction above will be the minimum point P_2, for then $x = IN = KI = c/\sqrt{3}$.

The intercept P_4 might be constructed by taking $K_4I = c$, but may be located (as in the figure) by laying off on IL, $IP_4 = c$.

If $k = a$, $KI = IL = IM = IN = a$. The corresponding point P_5 may be located by laying off IN_5 (which is also IL_5) $= a$, and Q_5P_5 through N_5, perpendicular to IL, and $= a$. [In any case, $y = IM + NQ = IM - QN$, so that P may always be located by measuring from Q a distance $y + QN = QP = IM$, parallel to the axis of ordinates.]

The Left Side.—The graph is symmetrical with regard to I, since $y \equiv f(x)$ is such that $f(-x_0) = -f(x_0)$; hence points determined on one side of I can be carried diametrically across to points on the other side. Points on the left side can also be determined directly by using negative values of k; that is, by locating K above I (K', not used in the construction, has the coördinates $(0, k)$).

Case (2). $\gamma - (\beta^2/3) > 0$. Write $\alpha = a^2$, $\gamma - (\beta^2/3) = c^2$; then the equation is

$$a^2y = x^3 + c^2x.$$

There is no extremum, and no intercept aside from the inflection I. $x = a^2/k$, $y = a^4/k^3 + c^2/k$ are parametric equations for the graph. If in the construction for Case (1), we make the slope of IQ c^2/a^2 instead of $-(c^2/a^2)$, the general description of the construction applies. [In either case, the slope of IQ is $[\gamma - (\beta^2/3)]$. The effect of the construction is to make the ordinate of P less than the ordinate of M in numerical value in Case (1), greater in numerical value in Case (2). The ordinate-difference, $QP = IM$, may be laid off directly from Q in either case.]

Case (3). $\gamma - (\beta^2/3) = 0$. The construction in Case (1) applies, though as in Case (2), there is no extremum, and no intercept except I. IQ coincides with IL, so that the ordinate of P is equal to the ordinate of M.

General Considerations. The Tangent at any Point.—The tangent at any point of the graph has for its y-intercept, $IT = -(2a^4/k^3) = -2IM$; for at any point $(a^2/k, a^4/k^3 \pm c^2/k)$ of $a^2y = x^3 \pm c^2x$, the slope is $3a^2/k^2 \pm c^2/a^2$, and the tangent itself has the equation

$$y = \left(\frac{3a^2}{k^2} \pm \frac{c^2}{a^2}\right)x - \frac{2a^4}{k^3}.$$

To draw the tangent at any point, P, that has been constructed, lay off $IT = -2IM$, join P and T; PT is the tangent.

The construction is shown for the tangents at P_1, P_3, P_4, P_5. T_2 coincides with R_2.

At diametrically opposite points, corresponding to equal and opposite values of k, the slope is the same; at two such points the tangents are parallel.

Graph for a Cubic Equation.—If the graph is to be used in connection with the solution of a cubic equation, the value of $\alpha = a^2$ is arbitrary, and may be so chosen as to separate or crowd the roots. The original axes may conveniently be located after the graph is drawn—with reference to the axes IL, IM, their intersection is at $(\beta/3, (1/27\alpha)(9\beta\gamma - 2\beta^3 - 27\delta))$. If it is convenient to make $a = c$, *i. e.*, $\alpha = \beta^2/3 - \gamma$, the construction will be somewhat simplified, as then $MR = IK$, and the point Q becomes superfluous.

The figure illustrates the construction for Case (1), showing seven points, with tangents, on each side of the inflection: the three special points P_2, P_4, P_5, a point in each of the segments IP_2 and P_2P_4, and two points beyond P_5.

In Case (2) or Case (3), fewer points would give an equally good guide for sketching the graph.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Dr. W. H. WILSON has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

Associate Professor W. B. FORD, of the University of Michigan, has been promoted to a full professorship of mathematics.

Miss PAULINE SPERRY, of Smith College, has accepted a position as instructor in mathematics at the University of California.

Assistant Professor L. A. RICE, of Syracuse University, has resigned to become instructor in mathematics at Tufts College.

Dr. IDA BARNEY, formerly instructor in Smith College, has been appointed professor of mathematics in Lake Erie College.

Professor W. A. GARRISON, of Union College, has been appointed professor of mathematics at King College, Bristol, Tennessee.

At the University of Nebraska, Miss L. RUNGE and Mr. A. BABBITT have been promoted to assistant professorships of mathematics.

Dr. A. W. HOBBS, of Johns Hopkins University, has been appointed instructor in mathematics at the University of North Carolina.

At the Southern Methodist University, Dallas, Texas, Associate Professor E. H. JONES has been promoted to a professorship of mathematics.

At the University of Toronto, Dr. SAMUEL BEATTY has been promoted to an assistant professorship of mathematics.

Dr. R. A. JOHNSON, of Adelbert College, Western Reserve University, has been appointed professor of mathematics at Hamline University, St. Paul, Minn.

At McPherson College, Kansas, Professor A. B. FRIZELL has resigned from the department of mathematics.

At Northwestern University, Dr. E. J. MOULTON has been promoted to an associate professorship of mathematics; and Mr. F. L. KERR and Mr. A. D. CAMPBELL have been appointed to instructorships in mathematics.

Professor S. F. NORRIS, of the department of mathematics in Baltimore City College, and a charter member of the Association, died on September 4, 1917.

Dr. M. O. TRIPP has resigned the professorship of mathematics at Olivet College, Olivet, Mich., and has accepted an assistant professorship at the University of Maine.

Professor A. D. BUTTERFIELD, head of the department of mathematics at Worcester Polytechnic Institute, has resigned to accept a commission in the aviation service of the U. S. government.

Dr. C. H. YEATON, assistant professor of mathematics at Northwestern University, and Mr. G. R. MIRICK, assistant in mathematics at the University of Michigan, have entered the national service, the former in connection with the Signal Corps at St. Louis, Mo.

At Louisiana State University, Assistant Professor S. T. SANDERS has been promoted to the headship of the department of mathematics, and Dr. I. C. NICHOLS has been appointed associate professor in the department.

Associate Professor R. P. BAKER has been made acting head of the department of mathematics at the University of Iowa, and Mr. R. E. GLEASON has been made instructor in mathematics.

Cornell University has granted leave of absence for the year 1917–18 to Dr. L. L. SILVERMAN, instructor in mathematics, for service upon the committee of public safety of the state of Massachusetts.

Mr. M. HEDLUND, for two years instructor in mathematics at Beloit College, was commissioned a second lieutenant in the Adjutant General's office at the close of the first officers' training school at Ft. Sheridan.

Professor H. E. HAWKES, of the department of mathematics of Columbia University, will be acting dean of Columbia College during the absence of Dean KEPPEL who is serving as confidential secretary to the Secretary of War.

At the University of Michigan, Mr. HERMAN BETZ, of Cornell University, has been appointed instructor in mathematics.

Mr. C. H. CLEVENGER, instructor in mechanics and mathematics at the University of Minnesota, has resigned to accept a position as assistant in an

investigation of stresses in chilled cast iron car wheels being conducted at the University of Illinois.

Dr. G. W. SMITH, formerly instructor in mathematics at the University of Colorado, has been appointed instructor in mathematics at Beloit College, Beloit, Wisconsin. Dr. SMITH did his graduate work at the University of Illinois, where he received the doctorate in June of this year.

The De Morgan medal, awarded every third year by the London Mathematical Society for eminent work in mathematics, has been awarded during the past ten years as follows: to Professor GLAZIER (1908), to Professor LAMB (1911), to Professor LARMOR (1914) and to Professor W. H. YOUNG (1917).

Professor J. F. MILLIS, who for a number of years has been head of the department of mathematics in the Parker School of Chicago, died at his home in Chicago on October 25, 1917, at the age of forty-two years. Professor MILLIS was the joint author of the well-known Stone-Millis series of elementary mathematical text-books.

Two charter members of the Association, Mr. O. S. ADAMS, and Mr. W. D. LAMBERT, of the Coast and Geodetic Survey, have been transferred to the War Department as first lieutenants and have entered the active service. Mr. P. C. PORTER, also a charter member, is enrolled in the army aviation ground-school at the University of Texas.

At the College for Women, Western Reserve University, Professor ANNA H. PALMIÉ has been granted leave of absence, her place being supplied by Dr. MARY F. CURTIS as instructor in mathematics. Mrs. W. E. BECKWITH has been promoted to an assistant professorship of mathematics.

During the summer Professor W. R. RANSOM, of Tufts College, had charge of Navigation Schools for the U. S. Shipping Board at Philadelphia. Professor D. T. WILSON, of Case School of Applied Science, gave instruction in the elements of navigation to men who are candidates for the merchant marine, and who have only elementary mathematical knowledge.

The feature of the opening number of Vol. XXIV of the *Bulletin* of the American Mathematical Society is a forty-six page paper by Professor G. A. BLISS, of the University of Chicago, on the "Integrals of Lebesgue," giving in detail the substance of Professor BLISS's symposium lecture presented at the April meeting of the Chicago Section of the Society.

At Purdue University, Professor T. G. ALFORD and Professor JACOB WESTLUND have retired from active service, the former on account of age, the latter on account of disability. Each has been granted a retiring allowance. Associate Professor WILLIAM MARSHALL has been promoted to a full professorship of mathematics, and instructor GLENN JAMES has been promoted to an assistant professorship.

Dr. RUTH E. GENTRY, formerly instructor in mathematics at Vassar College and a member of the American Mathematical Society, died at Indianapolis, Indiana, on October 15, 1917. Miss GENTRY was a graduate of the University of Michigan, received her doctorate at Byrn Mawr College, and as holder of a scholarship from the Association of Collegiate Alumnae she studied in Berlin and Paris.

The "*Guide to the Study and Use of Reference Books*," recently issued by the American Library Association, lists the following mathematical works: Encyclopédie des Sciences Mathématiques; Historical Introduction to Mathematical Literature, by G. A. MILLER; Mathematisches Vokabularium, by FELIX MÜLLER; Smithsonian Mathematical Tables; and Memorabilia Mathematica, by R. E. MORITZ.

Dr. J. D. MADDRILL, of the Travelers Insurance Company, and former director of the International Geodetic Observatory at Ukiah, Calif., and instructor in insurance mathematics at the University of California, has become actuary of the bureau of efficiency and economy at Washington. Among Dr. MADDRILL'S duties at Washington will be the preparation of a plan for pensioning the 300,000 civil employees of the government.

The Paris Academy of Sciences has awarded the following prizes in mechanics and mathematics: The Bordin prize of 3,000 francs to M. GASTON JULIA, now an officer in the French army; the Francoeur prize of 1,000 francs to M. HENRI VILLAT, for his publications on hydrodynamics; the Montyon prize of 700 francs to M. RENÉ DE SAUSSEURE, of Geneva, for his work in mechanics; and the Poncelet prize of 200 francs to M. JULES ANDRADE, professor at Besançon, for work in applied mechanics.

"A Sixth List of Writings on Determinants" by Sir THOMAS MUIR has appeared in Vol. XLVII, 1917, of *The Quarterly Journal of Pure and Applied Mathematics*. The five lists preceding the present one, also appearing in *The Quarterly*, cover the periods to 1880, 1885, 1900, 1905, 1910, respectively, and the sixth list covers the period from 1910–1915. The sixth list consists of titles and authors, date and place of publication, of four hundred twenty-four different publications.

At the University of Illinois, Dr. E. F. SIMONDS, of Columbia University, has been appointed instructor in mathematics; Mr. A. W. LARSEN has been appointed assistant in mathematics; and instructors Dr. C. M. HEBBERT and Dr. F. W. REED are teaching in the school of aviation. Dr. L. M. KELLS, instructor in mathematics, has entered the National Army at Camp Dodge.

At the University of Maine, Assistant Professor T. L. HAMLIN has resigned to accept a position in the department of mathematics at Union College, Schenectady, N. Y., and Mr. R. WOODS, instructor in mathematics, has resigned to do graduate work in the University of Illinois.

Baron KIKUCHI, the noted Japanese mathematician and scientist, died at his home in Japan on August 19, 1917, at the age of sixty-three years. His university training was obtained in England, where he was sent by the Japanese government. He entered Cambridge at the age of fifteen and was graduated among the wranglers at the age of twenty-two. Returning to Japan, he was immediately appointed professor of mathematics in the Imperial University, a position which he occupied for twelve years. Later he was dean of the College of Science, president of the Tokyo Imperial University, director of the Peer's School, president of the Kyoto Imperial University, and in 1901 he was appointed minister of education. In 1909–1910 Baron KIKUCHI visited England and the United States, giving lectures on Japanese education.

M. PAUL PAINLEVÉ, formerly professor of mathematics at the University of Paris and professor of mechanics at the Paris Polytechnic School, was chosen premier of the French Republic and served in that capacity until the recent overthrow of the French cabinet. While he was premier, the New York Evening Post made the following comment: "France has three of her most important posts in government and army filled by men especially renowned in mathematical proficiency. The Premier, M. PAINLEVÉ, knew enough at eleven and a half to have entitled him to his bachelor's degree, and later he attracted the attention of the mathematicians of Paris and was appointed to the professorships mentioned above. The commander-in-chief of the army, General PÉTAIN, now directing the gigantic struggle of the French armies, is a fine mathematician. And there is M. LOUCHEUR, the new minister of armaments, who gained especial mathematical distinction while a student at the Ecole Polytechnique.

The seventeenth annual meeting of the Central Association of Science and Mathematics Teachers will be held at Columbus, Ohio, on November 30 and December 1, 1917. At the opening general session Dr. W. O. THOMPSON, President of Ohio State University, will deliver an address on "Immediate and ultimate aims of science and mathematics teaching." At the mathematics section, Professor HARRIS HANCOCK, of the University of Cincinnati, will speak on "What course of study should be taken by a boy or girl in High School;" Mr. S. A. COURTIS, Supervisor of Educational Research, Detroit, Michigan, will speak on "Measurement of the products of teaching high school mathematics;" and Mr. J. A. FOBERG, of the Crane Junior College, Chicago, will present a report as chairman of the section committee on "mathematical requirements." Mr. Foberg is the representative of the Central Association delegated to act as a member of the National Committee on Mathematical Requirements now working under the auspices of the Mathematical Association of America.

The one hundred and ninety-third regular meeting of the American Mathematical Society was held in New York on October 27, 1917. Eleven papers appeared on the printed program. The eleventh regular meeting of the Southwestern Section will be held at the University of Oklahoma on December 1, 1917.

The seventieth meeting of the American Association for the Advancement of Science will be held at Pittsburgh during the week from December 28, 1917, to January 2, 1918. The chairman of Section A, Mathematics and Astronomy, is Professor H. N. RUSSELL of Princeton University, and the secretary is Professor F. R. MOULTON of the University of Chicago.

The following extract of a letter from Professor PAUL MONTEL of Paris to Professor Hedrick will be of interest to many of our readers:

"I received your letter of May 18th, and I want to thank you cordially for your kind expressions regarding my country. Our confidence was strengthened both morally and materially by the entry of the United States into the war, but no one here had any idea at first that this support would assume the formidable character which it now seems likely to take. I can assure you that the powerful effort which your country is now making fills us all with admiration. Yesterday I met the members of the scientific mission which has just returned from the United States: FABRY, ABRAHAM, etc. They are all elated over the warmth and the sympathy of the reception which they received.

"I desire to tell you that the Council of the French Mathematical Society has decided to accept articles written in either French, English or Italian. Hence whenever you send us an article, it will not be necessary to have it translated, unless you yourself desire."

NOTES ON THE ASSOCIATION.

The third annual meeting of the Mathematical Association of America will be held at the University of Chicago on Thursday and Friday, December 27–28, 1917, immediately preceding and in conjunction with the Winter meeting of the Chicago Section of the American Mathematical Society. There will be a joint session of the two organizations on Friday at which Professor W. B. Ford, of the University of Michigan, will deliver his retiring address as chairman of the Chicago Section on the subject: "A conspectus of the modern theory of divergent series." There will be a joint dinner on Thursday evening. The dinner was previously announced for Friday evening but was changed to Thursday in order to avoid conflict with the dinner of the Association of University Professors which occurs on Friday evening and which many of the mathematicians will desire to attend.

Arrangements for these meetings are in the hands of a joint committee consisting of W. D. Cairns of Oberlin College, Arnold Dresden of the University of Wisconsin, W. B. Ford of the University of Michigan, H. L. Rietz of the University of Illinois, J. A. Foberg of the Crane Junior College, Chicago, D. F. Campbell of Armour Institute, H. E. Cobb of Lewis Institute, E. J. Moulton of Northwestern University, and H. E. Slaught, chairman, of the University of Chicago.

The program of the Association meetings is in charge of a committee of three: Elizabeth B. Cowley of Vassar College, O. D. Kellogg of the University of Missouri, and E. J. Wilczynski, chairman, of the University of Chicago. Heretofore all places on the programs of the Association meetings have been filled by direct invitation of the Program Committee, but on this occasion it is proposed to devote one session to papers submitted by members on their own initiative. Abstracts of papers intended for this meeting should be submitted in form for publication to the Program Committee for approval and should be in the hands of the Chairman on or before Saturday, December 1, 1917. Some portion of the program will also be devoted to questions of interest to institutions, and all institutional members are invited to send delegates.

Programs and full announcements of arrangements for this meeting will be mailed to all members early in December.

In accordance with the action of the Council at Cleveland, President Cajori has appointed a committee of five to consider the feasibility of publishing a mathematical dictionary. The committee consists of R. C. Archibald of Brown University, H. L. Rietz of the University of Illinois, D. E. Smith of Columbia University, H. E. Slaught of the University of Chicago, and E. R. Hedrick, chairman, of the University of Missouri. The committee is expected to make at least a preliminary report at the annual meeting in Chicago.

Following up the suggestion of Professor Archibald in discussing undergraduate clubs at the Cleveland meeting (see October MONTHLY, page 357) a decision has been reached to form a special department of the MONTHLY to be devoted to undergraduate clubs. In the next January issue we should like to publish as complete a list of clubs as possible together with statistics as to membership and meetings. Will our readers, then, furnish us with such information as they have relative to existing clubs and their programs and membership for both this year and last year. Secretaries of clubs are particularly requested to furnish the desired information. Communications relative to this matter may be sent to Professor R. C. Archibald, Brown University, Providence, R. I.

President Cajori has announced that he proposes to adhere to the principle declared by Professor Hedrick last year, namely, that it is deemed to be for the best interests of the Association that the president should not be renominated for a second consecutive term. This announcement has already been made known to the members in connection with the preliminary ballot sent out about November first.

The Council has authorized the submission of certain amendments to the Constitution and By-Laws of the Association. These will be voted on at the annual meeting in Chicago, and in accordance with the constitution they are published herewith for comment or suggestion by members:

Article III, Section 2, to be modified to read as follows:

The President and Vice-Presidents shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms. The Secretary-Treasurer, and the Committee on Publications, consisting of the Manager, the Editor, and one other member, shall be appointed by the Council.

Article VI, Section 2, to be modified to read as follows:

The Council shall have full control of the publication and sale of the official journal.

By-Law 1, last sentence, to be modified to read:

Those who were admitted to membership before April 1, 1916, constitute the list of charter members.

By-Law 3, first sentence, to be replaced by the following:

Committees. The official journal shall be under the general management of the Committee on Publications. There shall also be appointed by the Council a Board of Associate Editors who shall give assistance in connection with the official journal under the direction of the Committee on Publications.

The above modifications are substantially as passed upon by the Council, except in the case of By-Law 1, which involves merely a change in tense of the verbs so as to conform to the present date.

The important changes in Article III of the Constitution pertain (1) to the Committee on Publications and (2) to the manner of choosing the Secretary-Treasurer. The reason for the first change is that the burdens of editorial work need to be further divided and distributed, and the reason for the second change is that the strictly business affairs of the Association need to be in the hands of an officer selected by the Council rather than elected by popular vote. All the other changes are consequences of these two.

The attention of institutional members is especially called to the report of the Committee on Libraries published in the October MONTHLY. A great amount of work has been expended by the committee in compiling this classified list and it is hoped that it may prove to be of real use, not only to institutional members, but also to all institutions of collegiate rank in the country, both as a means of checking the books now in their libraries and also as a stimulus to securing the books of this list not already possessed. In particular, it is hoped that this report may prove a real benefit to the mathematical departments of these institutions. The chairman invites correspondence on any matters pertaining to the suggestions contained in this report, and it is likely that a place may be provided on the program of the meeting for further discussion of it.

10

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CAVALIERI'S THEOREM IN HIS OWN WORDS.

By G. W. EVANS, Boston, Mass.

Introduction. Cavalieri's Theorem has recently come into some use in the teaching of elementary geometry as a means of unifying the ideas that lie at the basis of the mensuration of solids. The substance of the theorem is as follows:

If two solids have equivalent bases, and if sections parallel to the bases and equally distant from them in the two solids are also equivalent, then the solids are equivalent.

It is natural to think of the two solids referred to as being generated by two surfaces constantly equivalent to each other and constantly parallel to the respective bases, and we have an instinctive willingness to accept the statement that the two solids, so generated, are equivalent.

It is possible, however, to prove the theorem with as much rigor as is looked for in any demonstration involving the theory of limits; and it would be desirable, if it were possible, to prove this theorem at the beginning of the mensuration work of solid geometry. We then should be able to do away with the somewhat vexatious theorem about the equivalence of two parallelepipeds having equivalent bases and equal altitudes; and also to do away with the theorem about the equivalence of two pyramids having equivalent bases and equal altitudes. Moreover, the proof of the volume of a cylinder, or even of a cone, could be made, by means of Cavalieri's theorem, to depend directly upon the theorems in plane geometry about the area of a circle.

The more one examines the objects attainable by the use of this theorem as a basic proposition of mensuration, the more attractive it looks, especially in the matter of diminishing recourse to the theory of limits. On that account, the proof given by Cavalieri himself nearly three hundred years ago will doubtless be of interest to students and teachers of solid geometry. The diagram is a substantially exact copy of Cavalieri's own drawing, but the lettering has been